

BUTTERFLIES IN A SEMI-ABELIAN CONTEXT

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ABSTRACT. It is known that monoidal functors between internal groupoids in the category Grp of groups constitute the bicategory of fractions of the 2-category $Grpd(Grp)$ of internal groupoids, internal functors and internal natural transformations in Grp with respect to weak equivalences. Monoidal functors can be described equivalently by a kind of weak morphisms introduced by B. Noohi under the name of “butterflies”. In order to internalize monoidal functors in a wide context, we introduce the notion of internal butterflies between internal crossed modules in a semi-abelian category \mathcal{C} , and we show that they are morphisms of a bicategory $\mathcal{B}(\mathcal{C})$. Our main result states that, when in \mathcal{C} the notions of Huq commutator and Smith commutator coincide, then the bicategory $\mathcal{B}(\mathcal{C})$ of internal butterflies is the bicategory of fractions of $Grpd(\mathcal{C})$ with respect to weak equivalences (that is, internal functors which are internally fully faithful and essentially surjective on objects).

1. Introduction

A groupoid in the category of groups is a special case of strict monoidal category, tensor product being provided by the group structure on objects and arrows. Therefore, beyond internal functors, as arrows between groupoids in groups we can consider monoidal functors, that is functors between the underlying categories

$$F: \mathbb{H} \rightarrow \mathbb{G}$$

equipped with a natural and coherent family of isomorphisms

$$F^{x,y}: Fx + Fy \rightarrow F(x + y) \quad x, y \in H_0$$

Both notions of monoidal functor and internal functor are relevant as morphisms of groupoids in groups (just to cite an example, as special case of monoidal functors we get group extensions, whereas in the same case internal functors give split extensions, see Section 6), so the question of expressing in an internal way monoidal functors arises.

Three progresses have been recently accomplished in this direction. In [40] (see also [2]) B. Noohi has proved that the bicategory having groupoids in groups as objects and monoidal functors as 1-cells can be equivalently described using crossed modules of groups as objects and what he calls “butterflies” as arrows. Moreover, in a paper with E. Aldrovandi [2], the theory is pushed forward in order to include the more general situation

Financial support: FNRS grant 1.5.276.09 is gratefully acknowledged.

2000 Mathematics Subject Classification: 18D05, 18B40, 18E10, 18A40 .

Key words and phrases: bicategory of fractions, butterfly, internal groupoid, weak equivalence, semi-abelian category.

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where groups are replaced by internal groups in a (Grothendieck) topos. Noohi's butterflies (of [40]) has been studied in the case of Lie algebras on a field in [1], and in [42], where it is proved that butterflies between differential crossed modules (i.e. crossed modules of Lie algebras) represent homomorphisms of strict Lie 2-algebras.

On the other hand, in [49] it has been proved that the bicategory of groupoids in groups and monoidal functors is the bicategory of fractions of the bicategory of groupoids in groups and internal functors with respect to weak equivalences. Once again, the same result holds replacing groups with Lie algebras and monoidal functors with homomorphisms of strict Lie 2-algebras. In [23], M. Dupont has proved that butterflies provides the bicategory of fractions of internal functors with respect to weak equivalences when working internally to any abelian category.

The aim of this paper is to unify the results in [40], [1], [42], [23] and [49]. We introduce and study the bicategory $\mathcal{B}(\mathcal{C})$ of crossed modules and butterflies in a semi-abelian category \mathcal{C} . The main result is Theorem 5.8, where we prove that $\mathcal{B}(\mathcal{C})$ is the bicategory of fractions of the bicategory of groupoids and internal functors with respect to weak equivalences. This result gives a general answer to the specific problem recalled above: it describes weak internal functors that generalize at once monoidal functors in $\text{Grpd}(\text{Grp})$ and homomorphisms in $\text{Grpd}(\text{Lie})$, and works for other 2-dimensional algebraic settings, as for groupoids of Leibniz algebras, associative algebras, rings etc. The non pointed case will be examined in a forthcoming paper [39].

A few lines on the chosen context. We work internally to a semi-abelian category in which the notions of Huq commutator and Smith commutator coincide. This allows us, among other things, to use a simplified version of internal crossed modules without losing the equivalence with internal groupoids (see [37]). The categories of groups, Lie algebras, rings and many other algebraic structures not only in Set , but in any Grothendieck topos satisfy this condition (see Remark 9.4), so that our context include also that of [2].

Finally, let us give a glance to possible developments of the present work. Quite a lot of higher dimensional group theory has been developed starting from the pioneer works of P. Deligne [22] and A. Fröhlich and C.T.C. Wall [26] on Picard categories (also called 2-groups or categorical groups), taking monoidal functors as morphisms (see for example [48], [2], [24] and the references therein). On the other hand, group theory has been the paradigmatic example to develop in recent years semi-abelian categorical algebra (see Section 9 and the references therein). The fact of disposing of an internal notion of monoidal functor (the butterflies) should make possible to join these two generalizations of group theory and to develop a “higher dimensional semi-abelian categorical algebra” which could cover as special cases most of the known results on (strict) categorical groups and (strict) Lie 2-algebras.

The layout of this paper is as follows: in Section 2 we recall the equivalence between internal groupoids and internal crossed module, a result due to G. Janelidze (see [30]) and which holds in any semi-abelian category; in Sections 3 and 4 we study the bicategory $\mathcal{B}(\mathcal{C})$ of butterflies in a semi-abelian category \mathcal{C} with “Huq = Smith”; in Section 5 we prove that $\mathcal{B}(\mathcal{C})$ is the bicategory of fractions of internal functors with respect to

weak equivalences; in section 6 we examine the three leading examples of groups, rings and Lie algebras; section 7 is a short section devoted to the classification of extensions which follows from Section 5; in Section 8 we specialize the main result of Section 5 to the case where \mathcal{C} is a free exact category; finally, Section 9 is a reminder on protomodular and semi-abelian categories. The reader who is not familiar with semi-abelian categories should have a glance to Section 9 before reading Section 2.

Notation: the composite of $f: A \rightarrow B$ and $g: B \rightarrow C$ is written $f \cdot g$ or fg .

Terminology: bicategory means bicategory with invertible 2-cells.

2. Internal groupoids and internal crossed modules

An introduction to internal categories can be found in Chapter 8 of [8]. For basic facts on 2-categories and bicategories, see [5] or Chapter 7 in [8].

2.1. THE 2-CATEGORY OF INTERNAL GROUPOIDS. Let \mathcal{C} be a category with finite limits. We use the following notation:

- An (internal) groupoid \mathbb{G} in \mathcal{C} is depicted as

$$G_1 \times_{c,d} G_1 \xrightarrow{m} G_1 \begin{array}{c} \xrightarrow{d} \\ \xleftarrow{c} \end{array} G_0 \quad G_1 \xrightarrow{i} G_1$$

where

$$\begin{array}{ccc} G_1 \times_{c,d} G_1 & \xrightarrow{\pi_2} & G_1 \\ \pi_1 \downarrow & & \downarrow d \\ G_1 & \xrightarrow{c} & G_0 \end{array}$$

is a pullback;

- An (internal) functor $P = (p_1, p_0): \mathbb{G} \rightarrow \mathbb{H}$ is depicted as

$$\begin{array}{ccc} G_1 & \xrightarrow{p_1} & H_1 \\ d \downarrow \parallel c & & d \downarrow \parallel c \\ G_0 & \xrightarrow{p_0} & H_0 \end{array}$$

- An (internal) natural transformation $\alpha: P \Rightarrow Q: \mathbb{G} \rightarrow \mathbb{H}$ is depicted as

$$\begin{array}{ccc} G_1 & \xrightarrow{p_1} & H_1 \\ d \downarrow \parallel c & \nearrow \alpha & d \downarrow \parallel c \\ G_0 & \xrightarrow{p_0} & H_0 \end{array}$$

Internal groupoids, functors and natural transformations form a 2-category (with invertible 2-cells) denoted by $Grpd(\mathcal{C})$.

When dealing with internal structures, it is sometimes useful to use *virtual* objects and arrows as if those would be internal to the category of sets. For instance, we could describe the object $G_1 \times_{c,d} G_1$ as the “set” of composable arrows $\cdot \xrightarrow{f} \cdot \xrightarrow{g} \cdot$. Yoneda embedding makes this precise, as explained in [11], Metatheorem 0.2.7.

2.2. DISCRETE (CO)FIBRATIONS. An internal functor $P: \mathbb{G} \rightarrow \mathbb{H}$ as above is called a *discrete cofibration* when the commutative square $p_1 d = d p_0$ is a pullback. Dually, P is a *discrete fibration* when the square $p_1 c = c p_0$ is a pullback; for groupoids these two notions are equivalent.

Any groupoid comes with several canonical fibrations onto it, some of those being of interest in the rest of the paper.

For instance, let us consider the diagram

$$\begin{array}{ccc} R[c] & \xrightarrow{\tilde{d}} & G_1 \\ c_1 \downarrow & & \downarrow d \\ G_1 & \xrightarrow{d} & G_0 \end{array} \quad \begin{array}{c} c_2 \\ \downarrow \\ c \end{array}$$

where $(R[c], c_1, c_2)$ is a kernel pair of c , and \tilde{d} is the morphism that sends the pair of converging virtual arrows $x \xrightarrow{f} y \xleftarrow{g} z$ in the composition $x \xrightarrow{f \cdot g^{-1}} z$. The pair (\tilde{d}, d) is clearly a discrete fibration of groupoids. A similar argument can be developed for $R[d]$.

2.3. WEAK EQUIVALENCES. The following notion is due to M. Bunge and R. Paré, see [19]: An internal functor $P: \mathbb{G} \rightarrow \mathbb{H}$ is a weak equivalence if it is

- (internally) full and faithful, that is, the diagram

$$\begin{array}{ccccc} & & G_1 & & \\ & d \swarrow & \downarrow p_1 & \searrow c & \\ G_0 & & H_1 & & G_0 \\ & p_0 \searrow & \swarrow d & \searrow c & \swarrow p_0 \\ & & H_0 & & H_0 \end{array}$$

is a limit, and

- (internally) essentially surjective on objects, that is,

$$G_0 \times_{p_0, c} H_1 \xrightarrow{t_2} H_1 \xrightarrow{d} H_0$$

is a regular epimorphism, where

$$\begin{array}{ccc} G_0 \times_{p_0, c} H_1 & \xrightarrow{t_2} & H_1 \\ t_1 \downarrow & & \downarrow c \\ G_0 & \xrightarrow{p_0} & H_0 \end{array}$$

is a pullback.

Observe that $P: \mathbb{G} \rightarrow \mathbb{H}$ is (internally) full and faithful iff for every groupoid \mathbb{A} the functor

$$- \cdot P: \text{Grpd}(\mathcal{C})(\mathbb{A}, \mathbb{G}) \rightarrow \text{Grpd}(\mathcal{C})(\mathbb{A}, \mathbb{H})$$

is full and faithful in the usual sense. Moreover, a weak equivalence P is an equivalence iff

$$G_0 \times_{p_0, c} H_1 \xrightarrow{t_2} H_1 \xrightarrow{d} H_0$$

is a split epimorphism.

From now on we assume that \mathcal{C} is a semi-abelian category in which the implication “Huq \Rightarrow Smith” holds (for undefined notions and notations concerning semi-abelian categories the reader is addressed to Section 9).

2.4. INTERNAL CROSSED MODULES. An (internal) crossed module \mathbb{G} in \mathcal{C} is given by a morphism $\partial: G \rightarrow G_0$ and an action $\xi: G_0 \bowtie G \rightarrow G$ such that each part of the diagram

$$\begin{array}{ccc} G \bowtie G & \xrightarrow{\chi_G} & G \\ \partial \bowtie 1 \downarrow & & \downarrow 1 \\ G_0 \bowtie G & \xrightarrow{\xi} & G \\ 1 \bowtie \partial \downarrow & & \downarrow \partial \\ G_0 \bowtie G_0 & \xrightarrow{\chi_{G_0}} & G_0 \end{array}$$

commutes, χ_X being the canonical *conjugation action* for the object X . The commutativity of the upper part is called *Peiffer condition*, the commutativity of the lower part is called *precrossed module condition*.

A morphism $P: \mathbb{H} \rightarrow \mathbb{G}$ of crossed modules is given by morphisms $p: H \rightarrow G$ and $p_0: H_0 \rightarrow G_0$ such that each part of the diagram

$$\begin{array}{ccc} H_0 \bowtie H & \xrightarrow{p_0 \bowtie p} & G_0 \bowtie G \\ \xi \downarrow & & \downarrow \xi \\ H & \xrightarrow{p} & G \\ \partial \downarrow & & \downarrow \partial \\ H_0 & \xrightarrow{p_0} & G_0 \end{array}$$

commutes. In the following, we will refer to the upper commutative square above by saying that the pair (p, p_0) is *equivariant w.r.t. the actions*.

Internal crossed modules with their morphisms form a category denoted by $XMod(\mathcal{C})$.

2.5. REMARK. In an arbitrary semi-abelian category the notion of crossed module introduced by G. Janelidze in [30] is stronger than the one in 2.4. The notion we use (already considered in [30] and further studied in [38]) is equivalent to the original one thanks to the condition “Huq \Rightarrow Smith”, as proved in [37].

In the following proposition we consider $Grpd(\mathcal{C})$ as a category, that is, we forget natural transformations.

2.6. PROPOSITION. (*Janelidze [30]*) *The categories $Grpd(\mathcal{C})$ and $XMod(\mathcal{C})$ are equivalent.*

PROOF. (*sketch*) Let

$$G_1 \times_{c,d} G_1 \xrightarrow{m} G_1 \xrightleftharpoons[c]{d} G_0$$

be a groupoid and consider the following commutative diagram, where the rows are kernel diagrams,

$$\begin{array}{ccccc} G_0 \wr G & \xrightarrow{j_{G_0, G}} & G_0 + G & \xrightarrow{[1,0]} & G_0 \\ \exists! \downarrow \xi & & \downarrow [e, g] & & \downarrow 1 \\ G & \xrightarrow{g} & G_1 & \xrightarrow{c} & G_0 \end{array} \quad (1)$$

We obtain a crossed module

$$G \xrightarrow{g} G_1 \xrightarrow{d} G_0 \quad G_0 \wr G \xrightarrow{\xi} G$$

This describes the equivalence functor

$$J: Grpd(\mathcal{C}) \rightarrow XMod(\mathcal{C})$$

on objects; its extension to arrows is straightforward.

Conversely, let

$$G_0 \wr G \xrightarrow{\xi} G \xrightarrow{\partial} G_0$$

be a crossed module and consider the semi-direct product, given by the coequalizer

$$\begin{array}{ccc} G_0 \wr G & \xrightarrow{j_{G_0, G}} & G_0 + G \xrightarrow{q_\xi} G \rtimes_\xi G_0 \\ & \searrow \xi & \nearrow i_G \\ & G & \end{array}$$

We obtain a reflexive graph

$$G_1 = G \rtimes_{\xi} G_0 \begin{array}{c} \xrightarrow{d} \\ \xleftarrow{e} \\ \xleftarrow{c} \end{array} G_0$$

where c and e are, respectively, the canonical projection from and the canonical injection into the semi-direct product, and d is the unique morphism such that the diagram

$$\begin{array}{ccccc} G & \xrightarrow{g} & G \rtimes_{\xi} G_0 & \xleftarrow{e} & G_0 \\ & \searrow \partial & \downarrow d & \swarrow 1 & \\ & & G_0 & & \end{array}$$

commutes. For a detailed proof, see [30]. ■

We will refer to the functor J as to the *normalization functor*, and to its quasi-inverse as to the *denormalization functor*.

2.7. NOTATION. Here and in the following we will denote kernel of the codomain arrows with the lower case letter of the groupoid involved, e.g. the following sequence is exact:

$$G \xrightarrow{g} G_1 \xrightarrow{c} G_0$$

Moreover $g \cdot i = \ker d$ will be often denoted g^{\bullet} .

2.8. THE 2-CATEGORY OF CROSSED MODULES. The category $XMod(\mathcal{C})$ has an obvious 2-categorical structure. In fact it suffices to translate the notion of natural transformation for internal functors in the language of crossed module in order to obtain the 2-cells of $XMod(\mathcal{C})$.

2.9. DEFINITION. Consider two parallel morphisms $P, Q: \mathbb{H} \rightrightarrows \mathbb{G}$ of crossed modules. An arrow $\alpha: H_0 \rightarrow G_1 = G \rtimes_{\xi} G_0$ is a natural transformation between P and Q if $\alpha \cdot d = p_0$, $\alpha \cdot c = q_0$, and the diagram

$$\begin{array}{ccc} H & \xrightarrow{p} & G \\ \langle \partial \cdot \alpha, q \cdot g \rangle \downarrow & (\star) & \downarrow g \\ G_1 \times_{c,d} G_1 & \xrightarrow{m} & G_1 \end{array}$$

commutes.

The definition above involves several non-elementary constructions, such as the semidirect product $G_1 = G \rtimes G_0$ and the morphisms d and m . That is why we give an equivalent but easier to handle version of the diagram (\star) . This is done by the following

2.10. PROPOSITION. *Diagram (\star) of Definition 2.9 commutes iff the following one does:*

$$\begin{array}{ccc} H & \xrightarrow{\partial} & H_0 \\ \langle p, q \rangle \downarrow & & \downarrow \alpha \\ G \times G & \xrightarrow{m_0} & G_1 \end{array}$$

where $m_0 = g \sharp g^\bullet$ is the cooperator of the maps $G \xrightarrow{g} G_1 \xleftarrow{g^\bullet} G$ (see section 9.3).

For the proof, it suffices to compute with elements and then apply Yoneda embedding. To this end, recall that m_0 is the morphism that sends the pair of arrows $(x \xrightarrow{a} 0, y \xrightarrow{b} 0)$ to the composition $x \xrightarrow{a \cdot b^{-1}} y$, in other words m_0 performs a division a/b . Together with the conditions on domain and codomain, the last commutative diagram defines a transformation of *Peiffer graphs* [38].

2.11. LEMMA. *Arrows satisfying Definition 2.9 correspond biunivocally to natural transformations between the internal functors determined by the morphisms P and Q .*

PROOF. Recall (for instance, from [8]) that a natural transformation between two internal functors $P = (p_1, p_0)$ and $Q = (q_1, q_0)$: $\mathbb{H} \rightarrow \mathbb{G}$ is defined as a morphism $\alpha: H_0 \rightarrow G_1$ satisfying $\alpha \cdot d = p_0$, $\alpha \cdot c = q_0$, and such that the diagram

$$\begin{array}{ccc} H_1 & \xrightarrow{\langle p_1, c \cdot \alpha \rangle} & G_1 \times_{c, d} G_1 \\ \langle d \cdot \alpha, q_1 \rangle \downarrow & (\star\star) & \downarrow m \\ G_1 \times_{c, d} G_1 & \xrightarrow{m} & G_1 \end{array}$$

commutes. So we have to prove that (\star) commutes iff $(\star\star)$ commutes. The “if” part is dealt with by simply precomposing the diagram $(\star\star)$ with the monomorphism $h: H \rightarrow H_1$, conversely one observes that since the base category is protomodular, the pair

$$h, e: H_0 \rightarrow H_1$$

is (strongly) jointly epic, so that $(\star\star)$ commutes iff it commutes when precomposed both with h and e . The first precomposition is precisely (\star) , the second one is trivial. ■

In conclusion we have proved the following

2.12. COROLLARY. *The equivalence between $\text{Grpd}(\mathcal{C})$ and $X\text{Mod}(\mathcal{C})$ extends to a biequivalence.*

2.13. REMARK.

- (i) It has been proved in [25] that, in the equivalence $\text{Grpd}(\mathcal{C}) \simeq X\text{Mod}(\mathcal{C})$, a morphism $P: \mathbb{H} \rightarrow \mathbb{G}$ of crossed modules corresponds to a weak equivalence iff the arrows

induced on kernels and cokernels are isomorphisms:

$$\begin{array}{ccc}
 \ker \partial & \xrightarrow{\cong} & \ker \partial \\
 \downarrow & & \downarrow \\
 H & \xrightarrow{p} & G \\
 \downarrow \partial & & \downarrow \partial \\
 H_0 & \xrightarrow{p_0} & G_0 \\
 \downarrow & & \downarrow \\
 \operatorname{coker} \partial & \xrightarrow{\cong} & \operatorname{coker} \partial
 \end{array}$$

- (ii) Under the mentioned equivalence, it is easy to show that a morphism $P: \mathbb{H} \rightarrow \mathbb{G}$ of crossed modules corresponds to a discrete fibration iff $p: H \rightarrow G$ is an isomorphism.

2.14. LEMMA. *Let $(\partial: H \rightarrow H_0, \xi: H_0 \mathbin{\lhd} H \rightarrow H)$ be a crossed module and consider a morphism $\sigma: E \rightarrow H$. Consider also the pullback*

$$\begin{array}{ccc}
 E \times_{\sigma, \partial} H & \xrightarrow{\bar{\sigma}} & H \\
 \bar{\partial} \downarrow & (*) & \downarrow \partial \\
 E & \xrightarrow{\sigma} & H_0
 \end{array}$$

1. *The morphism $\bar{\partial}: E \times_{\sigma, \partial} H \rightarrow E$ can be equipped with a canonical action*

$$\bar{\xi}: E \mathbin{\lhd} (E \times_{\sigma, \partial} H) \rightarrow E \times_{\sigma, \partial} H$$

in such a way that the pair $(\bar{\partial}, \bar{\xi})$ is a crossed module and the diagram $()$ is a morphism of crossed modules.*

2. *Moreover, if $\sigma: E \rightarrow H_0$ is a regular epimorphism, then $(*)$ is a weak equivalence.*

PROOF. 1. By naturality of χ and the precrossed module condition on (∂, ξ) , the diagram

$$\begin{array}{ccccc}
 E \mathbin{\lhd} (E \times_{\sigma, \partial} H) & \xrightarrow{\sigma \mathbin{\lhd} \bar{\sigma}} & H_0 \mathbin{\lhd} H & \xrightarrow{\xi} & H \\
 1 \mathbin{\lhd} \bar{\partial} \downarrow & & & & \downarrow \partial \\
 E \mathbin{\lhd} E & \xrightarrow{\chi_E} & E & \xrightarrow{\sigma} & H_0
 \end{array}$$

commutes. Therefore, by the universal property of the pullback, we get a unique morphism

$$\bar{\xi}: E \mathbin{\lhd} (E \times_{\sigma, \partial} H) \rightarrow E \times_{\sigma, \partial} H$$

making the following diagram commute

$$\begin{array}{ccccc}
 E \wr E & \xleftarrow{1 \wr \bar{\partial}} & E \wr (E \times_{\sigma, \partial} H) & \xrightarrow{\sigma \wr \bar{\sigma}} & H_0 \wr H \\
 \chi_E \downarrow & (1) & \bar{\xi} \downarrow & (2) & \downarrow \xi \\
 E & \xleftarrow{\bar{\partial}} & E \times_{\sigma, \partial} H & \xrightarrow{\bar{\sigma}} & H .
 \end{array}$$

Since commutativity of (1) is the precrossed module condition on $(\bar{\partial}, \bar{\xi})$ and commutativity of (2) says that $(*)$ is a morphism of crossed modules, it remains to check Peiffer condition on $(\bar{\partial}, \bar{\xi})$, i.e., the commutativity of

$$\begin{array}{ccc}
 (E \times_{\sigma, \partial} H) \wr (E \times_{\sigma, \partial} H) & \xrightarrow{\chi} & E \times_{\sigma, \partial} H \\
 \bar{\partial} \wr 1 \downarrow & & \downarrow 1 \\
 E \wr (E \times_{\sigma, \partial} H) & \xrightarrow{\bar{\xi}} & E \times_{\sigma, \partial} H .
 \end{array}$$

For this, it suffices to compose with the pullback projections

$$E \xleftarrow{\bar{\partial}} E \times_{\sigma, \partial} H \xrightarrow{\bar{\sigma}} H .$$

The diagram obtained by composing with $\bar{\partial}$ commutes by naturality of χ and commutativity of (1), the one obtained by composing with $\bar{\sigma}$ commutes by naturality of χ , commutativity of (2) and Peiffer condition on (∂, ξ) .

2. Since $(*)$ is a pullback, kernels of parallel arrows are isomorphic. Henceforth, since σ and $\bar{\sigma}$ are regular epimorphisms, $(*)$ is also a pushout, so that the induced arrow $\text{coker } \bar{\partial} \rightarrow \text{coker } \partial$ is an isomorphism. From 2.13 we conclude that $(*)$ is a weak equivalence. \blacksquare

3. The bicategory of butterflies

We are going to describe the bicategory $\mathcal{B}(\mathcal{C})$ of crossed modules and butterflies in \mathcal{C} .

3.1. INTERNAL BUTTERFLIES. The following notion has been introduced, when \mathcal{C} is the category of groups, by B. Noohi in [40], see also [2] (a special case of butterflies was used by D. F. Holt in [28] to classify group extensions.). Let \mathbb{G} and \mathbb{H} be crossed modules. A (internal) butterfly from \mathbb{H} to \mathbb{G} is given by a commutative diagram of the form

$$\begin{array}{ccccc}
 H & & & & G \\
 & \searrow \kappa & & \swarrow \iota & \\
 & & E & & \\
 & \swarrow \sigma & & \searrow \rho & \\
 H_0 & & & & G_0
 \end{array}$$

such that

- i. $\kappa \cdot \rho = 0$, i.e. (κ, ρ) is a complex
- ii. $\iota = \ker \sigma$ and $\sigma = \text{coker } \iota$, i.e. (ι, σ) is an extension
- iii. the diagram

$$\begin{array}{ccccc} E \wr H & \xrightarrow{\sigma \wr 1} & H_0 \wr H & \xrightarrow{\xi} & H \\ \downarrow 1 \wr \kappa & & & & \downarrow \kappa \\ E \wr E & \xrightarrow{\chi_E} & & & E \end{array}$$

commutes, i.e. the pair $(\kappa, \sigma \wr 1 \cdot \xi)$ is a precrossed module,

- iv. the diagram

$$\begin{array}{ccccc} E \wr G & \xrightarrow{\rho \wr 1} & G_0 \wr G & \xrightarrow{\xi} & G \\ \downarrow 1 \wr \iota & & & & \downarrow \iota \\ E \wr E & \xrightarrow{\chi_E} & & & E \end{array}$$

commutes, i.e. the pair $(\iota, \rho \wr 1 \cdot \xi)$ is a precrossed module.

When no confusion arises, we denote a butterfly $(E, \kappa, \rho, \iota, \sigma)$ from \mathbb{H} to \mathbb{G} simply by

$$E: \mathbb{H} \rightarrow \mathbb{G}$$

Observe that since (ι, σ) is an exact sequence, the situation is not as symmetrical as it may appear at first sight. Actually ι is a mono, then the action $\rho \wr 1 \cdot \xi$ is nothing but the conjugation action χ_E restricted to the subobject G . Moreover ι can be recovered as the normalization of the equivalence relation $(R[\sigma], \sigma_1, \sigma_2)$, i.e. $\iota = \ker(\sigma_2) \cdot \sigma_1$.

Given butterflies $E, E': \mathbb{H} \rightarrow \mathbb{G}$, a morphism of butterflies is a morphism

$$f: E \rightarrow E'$$

such that the diagrams

$$\begin{array}{ccc} & E & \\ \kappa \nearrow & \downarrow f & \searrow \rho \\ H & & G_0 \\ \kappa' \searrow & & \nearrow \rho' \\ & E' & \end{array} \quad \begin{array}{ccc} & E & \\ \sigma \nearrow & \downarrow f & \searrow \iota \\ H_0 & & G \\ \sigma' \searrow & & \nearrow \iota' \\ & E' & \end{array}$$

commute. In particular, f is a morphism of extensions, so that by the short five lemma it is an isomorphism.

3.2. REMARK. Conditions (iii) and (iv) in 3.1 imply that the pairs $(\kappa, \sigma \flat 1 \cdot \xi)$ and $(\iota, \rho \flat 1 \cdot \xi)$ are indeed crossed modules and, therefore,

$$\begin{array}{ccc} H & \xleftarrow{1} & H \\ \partial \downarrow & & \downarrow \kappa \\ H_0 & \xleftarrow{\sigma} & E \end{array} \quad \text{and} \quad \begin{array}{ccc} G & \xrightarrow{1} & G \\ \iota \downarrow & & \downarrow \partial \\ E & \xrightarrow{\rho} & G_0 \end{array}$$

are morphisms of crossed modules, hence by 2.13 discrete fibrations.

3.3. FRACTORS. Using the equivalence between crossed modules and groupoids described in 2.6, butterflies correspond to *fractors*, i.e. diagrams of the form

$$\begin{array}{ccccc} & R & & R[\sigma] & \\ & \swarrow \bar{\sigma} & \searrow d & \swarrow \sigma_1 & \searrow \bar{\rho} \\ H_1 & & E & & G_1 \\ & \swarrow d & \searrow \sigma & \swarrow \rho & \searrow d \\ & H_0 & & G_0 & \end{array}$$

(Note: The above diagram is a simplified representation of the butterfly diagram. The actual diagram shows multiple arrows between nodes, representing the groupoid structure. Specifically, between R and E , there are arrows d and c . Between R and H_1 , there is an arrow $\bar{\sigma}$. Between R and $R[\sigma]$, there are arrows σ_1 and σ_2 . Between $R[\sigma]$ and G_1 , there is an arrow $\bar{\rho}$. Between H_1 and H_0 , there are arrows d and c . Between H_1 and E , there is an arrow σ . Between E and H_0 , there is an arrow σ . Between E and G_0 , there is an arrow ρ . Between E and G_1 , there is an arrow ρ . Between G_1 and G_0 , there are arrows d and c .)

where

1. functors $(\bar{\sigma}, \sigma)$ and $(\bar{\rho}, \rho)$ are discrete fibrations,
2. σ is a regular epimorphism, and $R[\sigma]$ is its kernel pair,
3. ρ coequalizes $d, c: R \rightrightarrows E$.

PROOF. Denormalizing the morphisms of crossed modules in the diagram

$$\begin{array}{ccccc} & H & & G & \\ & \swarrow 1 & \searrow \kappa & \swarrow \iota & \searrow 1 \\ H & & E & & G \\ & \swarrow \partial & \searrow \sigma & \swarrow \rho & \searrow \partial \\ & H_0 & & G_0 & \end{array}$$

one easily gets a fractor as above, where $H_1 = H \rtimes_{\xi} H_0$, $G_1 = G \rtimes_{\xi} G_0$ and $R = H \rtimes_{\sigma \flat 1 \cdot \xi} E$. The fact that the groupoid associated to ι is isomorphic to $(R[\sigma], \sigma_1, \sigma_2)$ is due to the fact that $\iota = \ker \sigma$. Finally, ρ coequalizes d and c since the pair

$$H \xrightarrow{\langle h, 0 \rangle} R \xleftarrow{e} E$$

is jointly (strongly) epic, by protomodularity.

Conversely, starting from a fractor as above, we get the butterfly

$$\begin{array}{ccccc}
 H & & & \ker \sigma \simeq G & \\
 \downarrow \partial & \searrow \langle h, 0 \rangle \cdot d & & \swarrow j & \downarrow \partial \\
 & & E & & \\
 & \swarrow \sigma & & \searrow \rho & \\
 H_0 & & & & G_0
 \end{array}$$

where $\langle h, 0 \rangle: H \rightarrow R$ comes from the universal property of the pullback

$$\begin{array}{ccc}
 H_1 & \xleftarrow{\bar{\sigma}} & R \\
 \downarrow c & & \downarrow c \\
 H_0 & \xleftarrow{\sigma} & E
 \end{array}$$

and the isomorphism $\ker \sigma \simeq G$ is the composite of the following isomorphisms determined by bottom pullback squares:

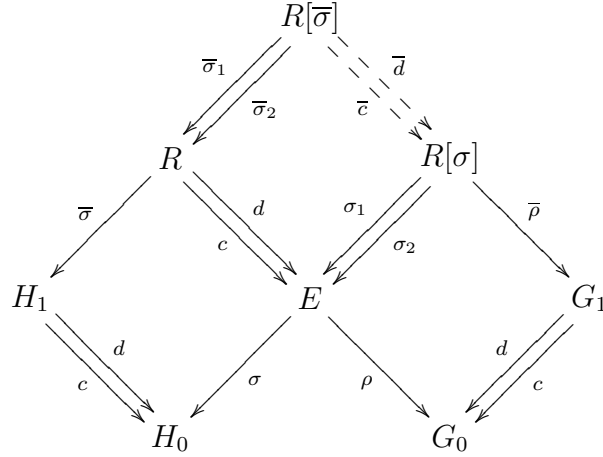
$$\begin{array}{ccccc}
 \ker \sigma & \xleftarrow{\simeq} & \ker \sigma_2 & \xrightarrow{\simeq} & G \\
 \downarrow j & & \downarrow & & \downarrow g \\
 E & \xleftarrow{\sigma_1} & R[\sigma] & \xrightarrow{\bar{\rho}} & G_1 \\
 \downarrow \sigma & & \downarrow \sigma_2 & & \downarrow c \\
 H_0 & \xleftarrow{\sigma} & E & \xrightarrow{\rho} & G_0
 \end{array}$$

■

3.4. **REMARK.** In a recent paper by D. Bourn [15], what we have called *fractor* is termed left regularly faithful profunctor

3.5. **REMARK.** Given a fractor as in 3.3, one can consider also the kernel pair of the map $\bar{\sigma}$ and perform the construction below, where dashed arrows are suitably obtained

by the universal property of the kernel pair $R[\sigma]$,



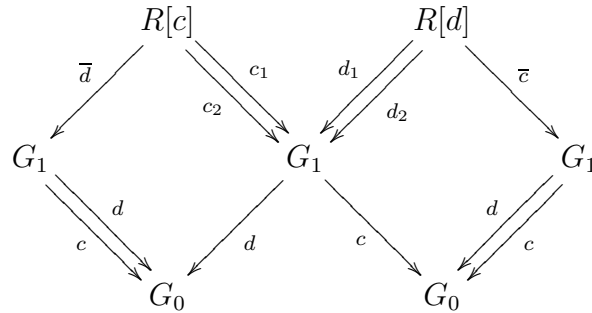
One finds out that the central square is a double groupoid over E . More precisely, it is a *centralizing double groupoid*, as defined by D. Bourn in [15], since $(\bar{\sigma}, \sigma)$ is a discrete fibration. Together with the two other squares, this gives rise to a particular profunctor $\mathbb{H} \bowtie \mathbb{G}$ of groupoids (profunctors were introduced by J. Bénabou with the name of *distributeurs* [6], the internal version can be found in [35]). The precise relationship between butterflies and profunctors will be described in a forthcoming paper [39].

3.6. IDENTITY BUTTERFLIES. We are going to prove that the canonical factor associated to a groupoid gives the identity butterfly associated to a crossed module.

Let \mathbb{G} be a crossed module. In order to construct the identity butterfly on \mathbb{G} , consider the groupoid associated to \mathbb{G} as in the proof of Proposition 2.6

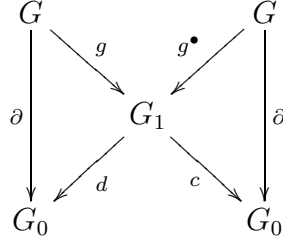
$$G_1 = G \rtimes_{\xi} G_0 \begin{array}{c} \xrightarrow{d} \\ \xleftarrow{c} \\ \xrightarrow{c} \\ \xleftarrow{d} \end{array} G_0$$

Then, following the example described in 2.2, one can associate \mathbb{G} to the fractor

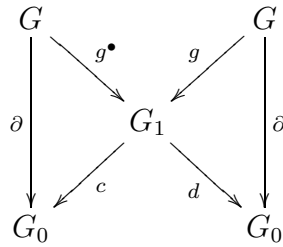


The butterfly associated to this fractor, by means of the normalization process described in 3.3, is called the identity butterfly of the crossed module \mathbb{G} . Actually it acts as an identity

w.r.t. the composition described in 3.7. It is represented explicitly in the diagram below:



Actually, in this paper, we will use as identity butterfly the isomorphic contravariant version of the one above:

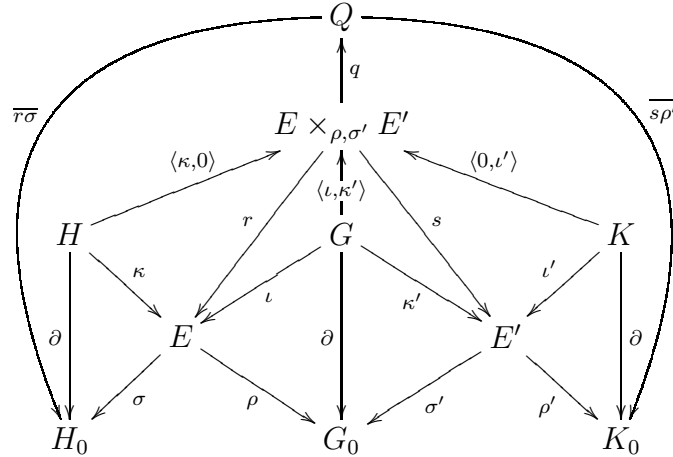


the isomorphism being realized by the inverse map $i: G_1 \rightarrow G_1$. This choice does not affect the computations (consider that the composition will be defined only up to isomorphisms), but is coherent with the normalization of a groupoid via the kernel of the codomain.

3.7. COMPOSITION OF BUTTERFLIES. Let

$$\mathbb{H} \xrightarrow{E} \mathbb{G} \xrightarrow{E'} \mathbb{K}$$

be butterflies. In order to construct their composition, consider the diagram



where

- $E \times_{\rho, \sigma'} E'$ is the pullback of ρ and σ' , with projections r and s , so that

$$\langle \kappa, 0 \rangle \cdot s = 0 = \langle 0, \iota' \rangle \cdot r, \quad \langle \kappa, 0 \rangle \cdot r = \kappa, \quad \langle 0, \iota' \rangle \cdot s = \iota', \quad \langle \iota, \kappa' \rangle \cdot r = \iota, \quad \langle \iota, \kappa' \rangle \cdot s = \kappa',$$

- (Q, q) is the cokernel of $\langle \iota, \kappa' \rangle$,
- $\overline{r\sigma}$ and $\overline{s\rho'}$ are defined by $q \cdot \overline{r\sigma} = r \cdot \sigma$, $q \cdot \overline{s\rho'} = s \cdot \rho'$.

The composition of E and E' is the butterfly

$$\begin{array}{ccccc}
 H & & & & K \\
 & \searrow \langle \kappa, 0 \rangle \cdot q & & \swarrow \langle 0, \iota' \rangle \cdot q & \\
 & & Q & & \\
 & \swarrow \overline{r\sigma} & & \searrow \overline{s\rho'} & \\
 H_0 & & & & K_0
 \end{array}$$

$\begin{array}{c} \downarrow \partial \qquad \qquad \downarrow \partial \end{array}$

PROOF. We have to check that the previous diagram is indeed a butterfly from \mathbb{H} to \mathbb{K} . Commutativity of wings and condition 3.1.i are easy to check.

Condition 3.1.ii: first observe that $\overline{r\sigma}$ is a regular epimorphism (because σ and σ' are), so that it is enough to show that $\langle 0, \iota' \rangle \cdot q$ is the kernel of $\overline{r\sigma}$. Since ι' is the kernel of σ' and r is a pullback of σ' , clearly $\langle 0, \iota' \rangle$ is the kernel of r . Consider now the commutative diagram

$$\begin{array}{ccccccc}
 G & \xleftarrow{1} & G & & & & \\
 \downarrow \iota & & \downarrow \langle \iota, \kappa' \rangle & & & & \\
 E & \xleftarrow{r} & E \times_{\rho, \sigma'} E' & \xleftarrow{\langle 0, \iota' \rangle} & K = \ker r & & \\
 \downarrow \sigma & & \downarrow q & & \downarrow f & & \\
 H_0 & \xleftarrow{\overline{r\sigma}} & Q & \xleftarrow{\quad} & \ker(\overline{r\sigma}) & &
 \end{array}$$

(\star)

where f is induced by the universal property of $\ker(\overline{r\sigma})$. It suffices to prove that (\star) is a pullback. Indeed, if (\star) is a pullback, then f is an isomorphism and, therefore, $\langle 0, \iota' \rangle \cdot q$ is the kernel of $\overline{r\sigma}$. Since q and σ are regular epimorphisms and ι is the kernel of σ , to show that (\star) is a pullback is equivalent to show that $\langle \iota, \kappa' \rangle$ is the kernel of q . Since $\langle \iota, \kappa' \rangle$ is a monomorphism (because ι is), to prove that $\langle \iota, \kappa' \rangle$ is a kernel (of its cokernel q) is equivalent to proving that $\langle \iota, \kappa' \rangle$ is closed under conjugation in $E \times_{\rho, \sigma'} E'$ (see [33, 38]). The action of $E \times_{\rho, \sigma'} E'$ on G is given by

$$(E \times_{\rho, \sigma'} E') \flat G \xrightarrow{r \flat 1} E \flat G \xrightarrow{\rho \flat 1} G_0 \flat G \xrightarrow{\xi} G$$

or, equivalently, by

$$(E \times_{\rho, \sigma'} E') \flat G \xrightarrow{s \flat 1} E' \flat G \xrightarrow{\sigma' \flat 1} G_0 \flat G \xrightarrow{\xi} G$$

Therefore, the normality of $\langle \iota, \kappa' \rangle$ in $E \times_{\rho, \sigma'} E'$ amounts to the commutativity of

$$\begin{array}{ccc}
 (E \times_{\rho, \sigma'} E') \flat G & \xrightarrow{r \flat 1} & E \flat G \xrightarrow{\rho \flat 1} G_0 \flat G \xrightarrow{\xi} G \\
 \downarrow 1 \flat \langle \iota, \kappa' \rangle & & \downarrow \langle \iota, \kappa' \rangle \\
 (E \times_{\rho, \sigma'} E') \flat (E \times_{\rho, \sigma'} E') & \xrightarrow{\chi} & E \times_{\rho, \sigma'} E'
 \end{array}$$

For this, compose with the pullback projections r and s and use the naturality of χ and, respectively, condition 3.1.iii on ι and condition 3.1.iv on κ' .

Condition 3.1.iii: since

$$q\flat 1: (E \times_{\rho, \sigma'} E') \flat H \rightarrow Q \flat H$$

is a (regular) epimorphism (see [38]), condition 3.1.iii follows from the commutativity of the whole diagram below

$$\begin{array}{ccccc}
 (E \times_{\rho, \sigma'} E') \flat H & \xrightarrow{q\flat 1} & Q \flat H & \xrightarrow{\overline{r\sigma}\flat 1} & H_0 \flat H & \xrightarrow{\xi} & H \\
 \downarrow 1\flat \langle \kappa, 0 \rangle & & & & & & \downarrow \langle \kappa, 0 \rangle \\
 (E \times_{\rho, \sigma'} E') \flat (E \times_{\rho, \sigma'} E') & \xrightarrow{\chi} & & & E \times_{\rho, \sigma'} E' & & \\
 \downarrow q\flat q & & & & \downarrow q & & \\
 Q \flat Q & \xrightarrow{\chi} & & & Q & &
 \end{array}$$

The lower region commutes by naturality of χ . For the commutativity of the upper region, compose with the pullback projections: composed with s , both paths go to zero; as far as r is concerned, use condition 3.1.iii on κ .

Condition 3.1.iv: same argument as for 3.1.iii. ■

3.8. PROPOSITION. *We have a bicategory*

$$\mathcal{B}(\mathcal{C})$$

with internal crossed modules as objects, butterflies as 1-cells, and morphisms of butterflies as 2-cells.

PROOF. Composition of butterflies and identity butterflies have been described in 3.7 and 3.6. The rest of the proof is long but straightforward. ■

Observe that in the identity butterfly (3.6) both diagonals are extensions. Butterflies with this property are called *flippable* (see [40]).

3.9. PROPOSITION. *A flippable butterfly $E: \mathbb{H} \rightarrow \mathbb{G}$ is an equivalence in the bicategory $\mathcal{B}(\mathcal{C})$. A quasi-inverse $E^*: \mathbb{G} \rightarrow \mathbb{H}$ is obtained by twisting the wings of E .*

PROOF. Keep in mind 3.7 and 3.6 and consider the diagram

$$\begin{array}{ccccc}
 & & R[\rho] & & \\
 & \nearrow r & \uparrow \langle \iota, \iota \rangle & \nwarrow s & \\
 & & G & & \\
 & \nwarrow \iota & & \nearrow \iota & \\
 E & & & & E \\
 \searrow \rho & & & & \swarrow \rho \\
 & & G_0 & &
 \end{array}$$

We have to prove that $H_1 = H \rtimes_{\xi} H_0$ is a cokernel of $\langle \iota, \iota \rangle$. Since (κ, ρ) is an extension, we can take (isomorphic) kernels in the left discrete fibration in the fractor corresponding to E (see 3.3)

$$\begin{array}{ccccc} H_1 & \xleftarrow{\bar{\sigma}} & R[\rho] & \xleftarrow{\langle \iota, \iota \rangle} & G \\ c \downarrow & & \rho_2 \downarrow & & \downarrow 1 \\ & & \rho_1 \downarrow & & \\ H_0 & \xleftarrow{\sigma} & E & \xleftarrow{\iota} & G \end{array}$$

Moreover, since σ and its pullback $\bar{\sigma}$ are regular epimorphisms, the horizontal rows are exact, and this concludes the proof. \blacksquare

4. Butterflies and morphisms of crossed modules

In order to prove that $\mathcal{B}(\mathcal{C})$ is the bicategory of fractions of $Grpd(\mathcal{C})$ with respect to weak equivalences (Theorem 5.8), we have to construct a homomorphism of bicategories

$$\mathcal{F}: Grpd(\mathcal{C}) \rightarrow \mathcal{B}(\mathcal{C})$$

This task will be completed only in section 5.6, since before we have to provide some necessary constructions.

A preliminary step consists in associating a split butterfly (Definition 4.1) to any morphism of crossed modules.

4.1. DEFINITION. A butterfly $E: \mathbb{H} \rightarrow \mathbb{G}$ is *split* when the extension

$$H_0 \xleftarrow{\sigma} E \xleftarrow{\iota} G$$

is split, that is, when there exists $s: H_0 \rightarrow E$ such that $s \cdot \sigma = 1_{H_0}$.

A morphism of split butterfly is simply a morphism of butterflies, so that it need not commute with sections.

4.2. FROM MORPHISMS TO SPLIT BUTTERFLIES. Let $P: \mathbb{H} \rightarrow \mathbb{G}$ be a morphism of crossed modules. We are going to construct a split butterfly $E_P: \mathbb{H} \rightarrow \mathbb{G}$.

Consider the pullback:

$$\begin{array}{ccc} E_P & \xrightarrow{\bar{p}} & G_1 \\ \sigma_P \downarrow & & \downarrow c \\ H_0 & \xrightarrow{p_0} & G_0 \end{array}$$

If $\xi: G_0 \mathbin{\text{b}} G \rightarrow G$ is the action corresponding to the split epi $c: G_1 \rightarrow G_0$, it is easy to show that

$$H_0 \mathbin{\text{b}} G \xrightarrow{p_0 \mathbin{\text{b}} 1} G_0 \mathbin{\text{b}} G \xrightarrow{\xi} G$$

is the action corresponding to the split epi $\sigma_P: E_P \rightarrow H_0$. We get the split butterfly $E_P: \mathbb{H} \rightarrow \mathbb{G}$:

$$\begin{array}{ccccc}
 H & & & & G \\
 \downarrow \partial & \searrow \langle \partial, pgi \rangle & & \swarrow \langle 0, g \rangle & \downarrow \partial \\
 & & E_P & & \\
 & \swarrow \sigma_P & & \searrow \bar{p}d & \\
 H_0 & & & & G_0
 \end{array}$$

PROOF. Commutativity of the wings is given by composing with pullback projections.

Condition 3.1.i: similarly one computes $\langle \partial, pgi \rangle \bar{p}d = pgi d = pgc = p0 = 0$.

Condition 3.1.ii: the North East - South West diagonal is a split extension, since it is the pullback of a split extension.

Condition 3.1.iii: To check the commutativity of

$$\begin{array}{ccccc}
 E_P \wr H & \xrightarrow{\sigma_P \wr 1} & H_0 \wr H & \xrightarrow{\xi} & H \\
 \downarrow 1 \wr \langle \partial, pgi \rangle & & & & \downarrow \langle \partial, pgi \rangle \\
 E_P \wr E_P & \xrightarrow{\chi_{E_P}} & & & E_P
 \end{array}$$

compose with the pullback projections $\sigma_P: E_P \rightarrow H_0$ and $\bar{p}: E_P \rightarrow G_1$. When composing with σ_P , use the naturality of χ and the precrossed module condition on \mathbb{H} . When composing with \bar{p} , the commutativity of the resulting diagram easily reduces to condition 3.1.iii on the identity butterfly on \mathbb{G} .

Condition 3.1.iv: to check the commutativity of

$$\begin{array}{ccccc}
 E_P \wr & \xrightarrow{\bar{p}d \wr 1} & G_0 \wr G & \xrightarrow{\xi} & G \\
 \downarrow 1 \wr \langle 0, g \rangle & & & & \downarrow \langle 0, g \rangle \\
 E_P \wr E_P & \xrightarrow{\chi_{E_P}} & & & E_P
 \end{array}$$

you can either compose once again with the pullback projections, or show that the induced action $\bar{p}d \wr 1 \cdot \xi$ is the unique that makes the subobject $(G, \langle 0, g \rangle)$ closed w.r.t. conjugation in E_P . ■

4.3. FROM SPLIT BUTTERFLIES TO MORPHISMS. We have just seen in 4.2 that every morphism $P: \mathbb{H} \rightarrow \mathbb{G}$ yields a split butterfly, namely E_P . Also the converse is true.

Indeed, let

$$\begin{array}{ccccc}
 H & & & & G \\
 \downarrow \partial & \searrow \kappa & & \swarrow \iota & \downarrow \partial \\
 & & E & & \\
 & \swarrow s & & \searrow \rho & \\
 H_0 & & & & G_0
 \end{array}$$

be a split butterfly. Precomposing the commutative diagram of condition 3.1.iii with $sb1: H_0 \bowtie H \rightarrow E \bowtie H$ we get the commutativity of

$$\begin{array}{ccc} H_0 \bowtie H & \xrightarrow{\xi} & H \\ sb\kappa \downarrow & & \downarrow \kappa \\ E \bowtie E & \xrightarrow{\chi_E} & E \end{array}$$

From the universal property of the semi-direct product (see [30], Theorem 1.3), we obtain a unique arrow $\bar{\kappa}$ making commutative the diagram

$$\begin{array}{ccccc} H & \xrightarrow{h} & H_1 & \xleftarrow{e} & H_0 \\ & \searrow \kappa & \downarrow \bar{\kappa} & \swarrow s & \\ & & E & & \end{array}$$

The requested morphism $P: \mathbb{H} \rightarrow \mathbb{G}$ is the one corresponding to the following internal functor (notation as in 3.3, Δ is the diagonal):

$$\begin{array}{ccccccc} H_1 & \xrightarrow{\bar{\kappa}} & E & \xrightarrow{\langle 1, \sigma s \rangle} & R[\sigma] & \xrightarrow{\bar{\rho}} & G_1 \xrightarrow{i} G_1 \\ \downarrow d \uparrow e \downarrow c & & \downarrow \sigma \uparrow s & & \downarrow \sigma_2 \uparrow \Delta & & \downarrow c \uparrow e \downarrow d \\ H_0 & \xrightarrow{1} & H_0 & \xrightarrow{s} & E & \xrightarrow{\rho} & G_0 \xrightarrow{1} G_0 \end{array}$$

(following [20], Proposition 2.1, it suffices to check that this is a morphism of reflexive graphs and, for this, use the dotted arrows).

4.4. REDUCED COMPOSITION. Given a morphism $Q: \mathbb{K} \rightarrow \mathbb{H}$ of crossed modules and a butterfly $E: \mathbb{H} \rightarrow \mathbb{G}$, we can transform Q into a split butterfly $E_Q: \mathbb{K} \rightarrow \mathbb{H}$ as in 4.2 and then to compose E_Q with E using composition of butterflies described in 3.7. We describe here a somehow easier way to calculate $E_Q \cdot E$ which will be called reduced composition and denoted by $Q \cdot_{rc} E$. Starting from

$$\begin{array}{ccccc} K & \xrightarrow{q} & H & & G \\ \downarrow \partial & & \downarrow \partial & \searrow \kappa & \downarrow \partial \\ & & & E & \\ & \swarrow \sigma & \swarrow \iota & & \\ K_0 & \xrightarrow{q_0} & H_0 & & G_0 \end{array}$$

consider the pullback

$$\begin{array}{ccc} E' & \xrightarrow{q'} & E \\ \sigma' \downarrow & & \downarrow \sigma \\ K_0 & \xrightarrow{q_0} & H_0 \end{array} \quad (2)$$

and the arrows

$$\langle 0, \iota \rangle: G \rightarrow E' \quad \langle \partial, p \cdot \kappa \rangle: K \rightarrow E'.$$

The $Q \cdot_{rc} E$ is given by

$$\begin{array}{ccccc} K & & & & G \\ & \searrow \langle \partial, p \cdot \kappa \rangle & & \swarrow \langle 0, \iota \rangle & \\ & & E' & & \\ & \swarrow \sigma' & & \searrow q' \cdot \rho & \\ K_0 & & & & G_0 \end{array}$$

and it coincides with the butterfly $E_Q \cdot_{rc} E$. In particular, if $I_{\mathbb{H}}: \mathbb{H} \rightarrow \mathbb{H}$ is the identity butterfly (3.6), then $Q \cdot_{rc} I_{\mathbb{H}}$ is precisely the split butterfly E_Q as in 4.2.

PROOF. We have to prove that $E_Q \cdot E = Q \cdot_{rc} E$. Let us consider the following picture, where all the squares are pullbacks and moreover, down-right square is the discrete fibration of 3.5:

$$\begin{array}{ccccc} & & K_0 \times_{q,c} H_1 \times_{d,\sigma} E & & \\ & \swarrow r & & \searrow \phi & \\ & E_Q & & R & \\ \sigma_Q \swarrow & & \bar{q} \searrow & \bar{\sigma} \swarrow & \bar{c} \searrow \\ K_0 & & H_1 & & E \\ q_0 \searrow & & c \swarrow & d \searrow & \sigma \swarrow \\ & H_0 & & H_0 & \end{array}$$

By commutativity of limits, the topmost object is the limit over the w -shaped diagram $\{q_0, c, d, \sigma\}$, whence the notation adopted. The pullback (2) determines a unique $\omega: K_0 \times_{q,c} H_1 \times_{d,\sigma} E \rightarrow E' = K_0 \times_{q,\sigma} E$, such that $\omega q' = \phi \bar{c}$ and $\omega \sigma' = r \sigma_Q$. Now we can consider the diagram

$$\begin{array}{ccccccc} G & \xleftarrow{1} & G & & & & \\ \langle 0, \iota \rangle \downarrow & & \downarrow \langle 0, 0, \iota \rangle & & & & \\ E' & \xleftarrow{\omega} & K_0 \times_{q,c} H_1 \times_{d,\sigma} E & \xleftarrow{\langle 0, h, \kappa \rangle} & H & & \\ \sigma' \downarrow & & \downarrow r & & \downarrow 1 & & \\ K_0 & \xleftarrow{\sigma_Q} & E_Q & \xleftarrow{\langle 0, h \rangle} & H & & \end{array}$$

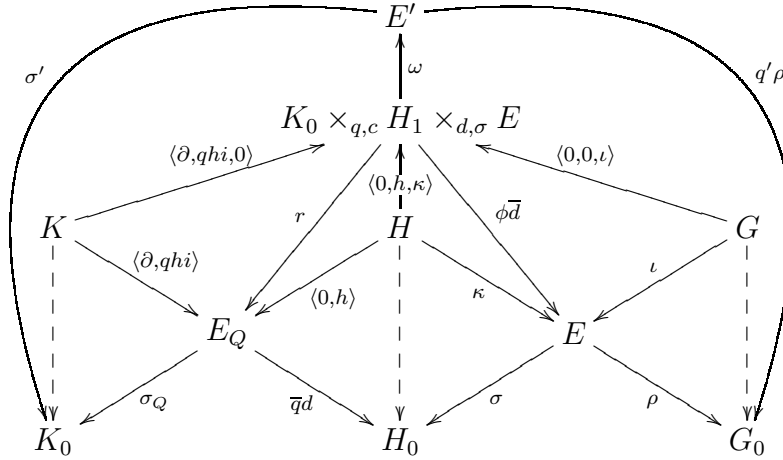
By composing with pullback projections, one easily shows that (i) and (iii) commute, so that all the squares are commutative. Then, since r is a regular epimorphism, by

the (normalized) Barr-Kock Theorem 9.2, (ii) is a pullback square, hence ω is a regular epimorphism and it has the same kernel as σ_Q . Moreover, since $\ker(\sigma_Q) = \langle 0, h \rangle$, (iii) proves that $\ker(\omega) = \langle 0, h, \kappa \rangle$

So far, we proved a technical

4.5. LEMMA. *The sequence $H \xrightarrow{\langle 0, h, \kappa \rangle} K_0 \times_{q,c} H_1 \times_{d,\sigma} E \xrightarrow{\omega} E'$ is exact.*

Now we can finally prove reduced composition. To this end, let us consider the following diagram



The two butterflies involved are (from left to right), the split butterfly $E_Q: \mathbb{K} \rightarrow \mathbb{H}$ corresponding to the morphism Q , and $E: \mathbb{H} \rightarrow \mathbb{G}$. What we are to show is that the above diagram yields the composition of the two. In fact the resulting butterfly would be precisely $Q \cdot_{rc} E$, as desired.

By composition of pullbacks, the square $r \cdot \bar{q}d = \phi \bar{d} \cdot \sigma$ above is a pullback, and by Lemma 4.5 ω is the cokernel of $\langle 0, h, \kappa \rangle$. Moreover σ' is (the only morphism) such that $\omega \sigma' = r \sigma_Q$ and $q' \rho$ is (the only one) such that $\omega q' \rho = \phi \bar{d} \rho$, and this concludes the proof. ■

The following statement will help us in defining the embedding of crossed modules into butterflies.

4.6. PROPOSITION. *Reduced composition gives yields on hom-categories a left monoidal action of crossed module morphisms on butterflies, i.e. the following formulæ coherently hold when well defined, for morphisms P and Q and for butterflies E and F :*

$$A1. \quad Q \cdot_{rc} EF \cong (Q \cdot_{rc} E)F$$

$$A2. \quad PQ \cdot_{rc} E \cong P \cdot_{rc} (Q \cdot_{rc} E)$$

$$A3. \quad I \cdot_{rc} E \cong E$$

PROOF. (outline) The proof of A3 is trivial, that of A2 is straightforward. The proof of A1 can be easily deduced from the particular case

$$A1^*: Q \cdot_{rc} F \cong (Q \cdot_{rc} I)F$$

where I is the identity butterfly on the domain of F . Actually one computes

$$Q \cdot_{rc} EF \cong (Q \cdot_{rc} I)EF \cong ((Q \cdot_{rc} I)E)F \cong ((Q \cdot_{rc} (IE))F \cong (Q \cdot_{rc} E)F$$

Hence we are to prove $A1^*$ holds, but since $Q \cdot_{rc} I = E_Q$, this is precisely the content of the proof of the consistency of reduced composition described above. ■

5. Butterflies are fractions

In this section we prove the main result of the paper, but first it is necessary to introduce the fractions whose the title refers to. As for the case of groups (see [40]), given a butterfly it is possible to construct a span of morphisms, one being a weak equivalence. By denormalizing, this yields a fraction of internal functors.

Categories of fractions have been introduced by P. Gabriel and M. Zisman in [27] to give a simplicial construction of the homotopy category of CW complexes. In order to study toposes locally equivalent to toposes of sheaves on a topological space, in [44] D. Pronk generalized Gabriel-Zisman concept introducing bicategories of fractions.

5.1. BICATEGORIES OF FRACTIONS. Imitating the usual universal property of the category of fractions, it is clear how to state the universal property of the bicategory of fractions

$$\mathcal{P}_\Sigma: \mathcal{B} \rightarrow \mathcal{B}[\Sigma^{-1}]$$

of a bicategory \mathcal{B} with respect to a class Σ of 1-cells ([44]): the bicategory of fractions of \mathcal{B} with respect to Σ is a homomorphism of bicategories

$$\mathcal{P}_\Sigma: \mathcal{B} \rightarrow \mathcal{B}[\Sigma^{-1}]$$

universal among all homomorphisms $\mathcal{F}: \mathcal{B} \rightarrow \mathcal{A}$ such that $\mathcal{F}(S)$ is an equivalence for all $S \in \Sigma$. This means that, for every bicategory \mathcal{A} ,

$$\mathcal{P}_\Sigma \cdot - : \text{Hom}(\mathcal{B}[\Sigma^{-1}], \mathcal{A}) \rightarrow \text{Hom}_\Sigma(\mathcal{B}, \mathcal{A})$$

is a biequivalence of bicategories, where a homomorphism $\mathcal{F}: \mathcal{B} \rightarrow \mathcal{A}$ lies in $\text{Hom}_\Sigma(\mathcal{B}, \mathcal{A})$ when $\mathcal{F}(S)$ is an equivalence for all $S \in \Sigma$.

The real challenge with bicategories of fractions is to find an explicit, maniable description of $\mathcal{B}[\Sigma^{-1}]$. A first general result in this direction, established in [44], states that if Σ satisfies some suitable conditions (has a “right calculus of fractions”) then the bicategory of fraction exists and can be described as follows: the objects of $\mathcal{B}[\Sigma^{-1}]$ are those

of \mathcal{B} and the 1-cells of $\mathcal{B}[\Sigma^{-1}]$ are spans of 1-cells in \mathcal{B} with the backward leg in Σ (this is the non straightforward generalization of a well-known result from [27]).

In order to prove that butterflies provide the bicategory of fractions of $\text{Grpd}(\mathcal{C})$ with respect to weak equivalences, we will use the following result.

5.2. PROPOSITION. (*Pronk [44]*) *Let Σ be a class of 1-cells in a bicategory \mathcal{B} . Assume that Σ has a right calculus of fractions and consider a homomorphism of bicategories $\mathcal{F}: \mathcal{B} \rightarrow \mathcal{A}$ such that*

EF0. $\mathcal{F}(S)$ is an equivalence for all $S \in \Sigma$;

EF1. \mathcal{F} is surjective up to equivalence on objects;

EF2. \mathcal{F} is full and faithful on 2-cells;

EF3. For every 1-cell F in \mathcal{A} there exist 1-cells G and W in \mathcal{B} with W in Σ and a 2-cell $\mathcal{F}(G) \Rightarrow \mathcal{F}(W) \cdot F$.

Then the (essentially unique) extension

$$\widehat{\mathcal{F}}: \mathcal{B}[\Sigma^{-1}] \rightarrow \mathcal{A}$$

of \mathcal{F} through \mathcal{P}_Σ is a biequivalence.

5.3. THE MORPHISMS κ AND ι COOPERATE. Before we show how a butterfly turns into a fraction, we need one more property of butterflies.

Consider a butterfly $E: \mathbb{H} \rightarrow \mathbb{G}$. The arrows

$$\kappa: H \rightarrow E \leftarrow G: \iota$$

cooperate (see section 9.3), that is, there exists a unique arrow $\varphi = \kappa \sharp \iota$ such that the diagram

$$\begin{array}{ccccc} H & \xrightarrow{\langle 1,0 \rangle} & H \times G & \xleftarrow{\langle 0,1 \rangle} & G \\ & \searrow \kappa & \downarrow \varphi & \swarrow \iota & \\ & & E & & \end{array}$$

commutes. Indeed, the fact that κ and ι cooperate is equivalent to the fact that the composition

$$G \diamond H \xrightarrow{\delta} G + H \xrightarrow{[\iota, \kappa]} E$$

is the zero morphism (see for example [38]), where δ is the diagonal of the pullback

$$\begin{array}{ccc} G \diamond H & \xrightarrow{\delta_1} & G \flat H \\ \delta_2 \downarrow & & \downarrow j_{G,H} \\ H \flat G & \xrightarrow{j_{H,G}} & G + H \end{array}$$

The equation $\delta \cdot [\iota, \kappa] = 0$ follows from the commutativity of

$$\begin{array}{ccccc}
 G \diamond H & \xrightarrow{\delta_2} & H \flat G & \xrightarrow{j_{H,G}} & G + H & \xrightarrow{[0,1]} & H \simeq 0 \flat H \\
 \downarrow \delta & & & & & & \downarrow !\flat 1 \\
 & & & & & & H_0 \flat H \\
 & & & & & & \downarrow \xi \\
 & & & & & & H \\
 & & & & & & \downarrow \kappa \\
 G + H & \xrightarrow{[\iota, \kappa]} & & & & & E
 \end{array}$$

which can be reduced to the commutativity of

$$\begin{array}{ccccc}
 E \flat H & \xrightarrow{\sigma \flat 1} & H_0 \flat H & \xrightarrow{\xi} & H \\
 \downarrow j_{E,H} & & & & \downarrow \kappa \\
 E + H & \xrightarrow{[1, \kappa]} & & & E
 \end{array}$$

Finally, this is a consequence of condition 3.1.iii using that

$$\chi_E = j_{E,E} \cdot [1, 1]: E \flat E \rightarrow E + E \rightarrow E$$

5.4. **REMARK.** Observe that the fact that κ and ι cooperate may be used as a starting point for creating many non-trivial examples of butterfly: one starts by considering two cooperating normal subobjects and then computes their respective cokernels.

5.5. **SPAN ASSOCIATED TO A BUTTERFLY.** So far we established that the two crossed modules κ and ι cooperate. Now, we are going to prove that φ is itself a crossed module, for a suitable action $\bar{\xi}$, and that the diagram

$$\begin{array}{ccccc}
 H & \xleftarrow{\pi_H} & H \times G & \xrightarrow{\pi_G} & G \\
 \partial \downarrow & (1) & \varphi \downarrow & (2) & \downarrow \partial \\
 H_0 & \xleftarrow{\sigma} & E & \xrightarrow{\rho} & G_0
 \end{array}$$

is a span of crossed modules,

$$\mathbb{H} \xleftarrow{(\pi_H, \sigma)} [E] \xrightarrow{(\pi_G, \rho)} \mathbb{G}$$

with (1) being a weak equivalence.

PROOF. The commutativity of (1) and (2) can be proved by precomposing with the jointly epimorphic pair

$$\langle 1, 0 \rangle: H \rightarrow H \times G \leftarrow G: \langle 0, 1 \rangle$$

Moreover, (1) is a pullback because it is commutative and the regular epimorphisms π_H and σ have same kernel G (use 9.2). Therefore, we can apply Lemma 2.14 to (1) getting that φ is a crossed module and that (1) is a weak equivalence of crossed modules. The action $\bar{\xi}$ that makes φ a crossed module is the unique morphism such that $\bar{\xi}\pi_H = (\sigma\flat\pi_H)\xi$ and $\bar{\xi}\varphi = (1\flat\varphi)\chi_E$, (see section 2.14).

It remains to show that (2) is a morphism of crossed modules, i.e. that the diagram

$$\begin{array}{ccc} E\flat(H \times G) & \xrightarrow{\rho\flat\pi_G} & G_0\flat G \\ \bar{\xi} \downarrow & (3) & \downarrow \xi \\ H \times G & \xrightarrow{\pi_G} & G \end{array}$$

commutes. For this, we need a different description of $\bar{\xi}$.

Observe that the pullback (1) can be expressed as composite of two pullbacks using the discrete fibration associated to the left wing of the butterfly $E: \mathbb{H} \rightarrow \mathbb{G}$ (see 3.3):

$$\begin{array}{ccc} H \times G & \xrightarrow{\pi_H} & H \\ \bar{h} \downarrow & & \downarrow h \\ R & \xrightarrow{\bar{\sigma}} & H_1 \\ d \downarrow & & \downarrow d \\ E & \xrightarrow{\sigma} & H_0 \end{array} \quad \begin{array}{c} \varphi \quad \quad \quad \partial \end{array}$$

Moreover, since h is the kernel of $c: H_1 \rightarrow H_0$ and ι is the kernel of $\sigma: E \rightarrow H_0$, comparing the diagrams

$$\begin{array}{ccc} H \times G & \xrightarrow{\pi_H} & H \\ \bar{h} \downarrow & & \downarrow h \\ R & \xrightarrow{\bar{\sigma}} & H_1 \\ c \downarrow & & \downarrow c \\ E & \xrightarrow{\sigma} & H_0 \end{array} \quad \text{and} \quad \begin{array}{ccc} H \times G & \xrightarrow{\pi_H} & H \\ \pi_G \downarrow & & \downarrow \\ G & \longrightarrow & 0 \\ \iota \downarrow & & \downarrow \\ E & \xrightarrow{\sigma} & H_0 \end{array}$$

(where all squares are pullbacks) we get the commutativity of

$$\begin{array}{ccc} H \times G & \xrightarrow{\pi_G} & G \\ \bar{h} \downarrow & & \downarrow \iota \\ R & \xrightarrow{c} & E \end{array}$$

Observe now that since $h: H \rightarrow H_1$ is a normal mono, there exists a unique $\chi_! : H_1 \bowtie H \rightarrow H$ such that

$$\begin{array}{ccc} H_1 \bowtie H & \xrightarrow{\chi_!} & H \\ 1 \bowtie h \downarrow & & \downarrow h \\ H_1 \bowtie H_1 & \xrightarrow{\chi} & H_1 \end{array}$$

commutes. From this fact, it follows easily that also

$$\begin{array}{ccc} R \bowtie (H \times G) & \xrightarrow{(\bar{\sigma} \bowtie \pi_H) \chi_!} & H \\ (1 \bowtie \bar{h}) \chi \downarrow & & \downarrow h \\ R & \xrightarrow{\bar{\sigma}} & H_1 \end{array}$$

commutes. By the universal property of the pullback of $\bar{\sigma}$ and h we get a unique morphism x such that

$$\begin{array}{ccccc} R \bowtie R & \xleftarrow{1 \bowtie \bar{h}} & R \bowtie (H \times G) & \xrightarrow{\bar{\sigma} \bowtie \pi_H} & H_1 \bowtie H \\ \chi \downarrow & & x \downarrow & & \downarrow \chi_! \\ R & \xleftarrow{\bar{h}} & H \times G & \xrightarrow{\pi_H} & H \end{array}$$

commutes. The action $\bar{\xi}$ factorizes through x as follows:

$$\begin{array}{ccc} E \bowtie (H \times G) & \xrightarrow{\bar{\xi}} & H \times G \\ & \searrow e \bowtie 1 & \nearrow x \\ & R \bowtie (H \times G) & \end{array}$$

To check the commutativity of the previous triangle, compose with the pullback projections

$$E \xleftarrow{\varphi} H \times G \xrightarrow{\pi_H} H$$

When composing with φ , use that $\varphi = \bar{h}d$ and the left-hand square in the definition of x . When composing with π_H , use the right-hand square in the definition of x and the commutativity of

$$\begin{array}{ccc} H_0 \bowtie H & \xrightarrow{\xi} & H \\ & \searrow e \bowtie 1 & \nearrow \chi_! \\ & H_1 \bowtie H & \end{array}$$

(this last equation is easy to verify: compose with the monomorphism h and use the definition of χ_{H_1} , see diagram (1) of Proposition 2.6).

We are ready to prove the commutativity of diagram (3): replace $\bar{\xi}$ by $(e \bowtie 1)x$, compose with the monomorphism $\iota: G \rightarrow E$ and use the left-hand square in the definition of x , condition 3.1.iv and the equation $\pi_{G\iota} = \bar{h}c$ established above. ■

5.6. THE UNIVERSAL HOMOMORPHISM. Combining the equivalence

$$J: \text{Grpd}(\mathcal{C}) \rightarrow \text{XMod}(\mathcal{C})$$

of Proposition 2.6 with the construction of the split butterfly E_P associated with a morphism P (4.2), we are ready to define a homomorphism of bicategories

$$\mathcal{F}: \text{Grpd}(\mathcal{C}) \rightarrow \mathcal{B}(\mathcal{C}).$$

On objects and on 1-cells we let

$$\mathcal{F}(\mathbb{H}) = J(\mathbb{H}), \quad \mathcal{F}(P: \mathbb{H} \rightarrow \mathbb{G}) = (E_{J(P)}: J(\mathbb{H}) \rightarrow J(\mathbb{G})).$$

The composition and the identity structural isomorphisms are defined by using the properties described in Proposition 4.6, by identifying the behavior of \mathcal{F} on 1-cells with the action of the (reduced) composition with the identity butterfly (see Remark 5.7).

It remains to define \mathcal{F} on 2-cells. Let $\alpha: P \Rightarrow Q: \mathbb{H} \rightarrow \mathbb{G}$ be a natural transformation; there exists a unique morphism $\bar{\alpha}$ such that the diagram

$$\begin{array}{ccccc} E_{J(P)} & \xrightarrow{\sigma_P} & H_0 & & \\ & \searrow \bar{\alpha} & \searrow \alpha & & \\ & & G_1 \times_{c,d} G_1 & \xrightarrow{\pi_2} & G_1 \\ & \searrow \bar{p} & \downarrow \pi_1 & & \downarrow d \\ & & G_1 & \xrightarrow{c} & G_0 \end{array}$$

commutes. Finally, $\mathcal{F}(\alpha): E_{J(P)} \rightarrow E_{J(Q)}$ is the unique morphism such that the diagram

$$\begin{array}{ccccc} E_{J(P)} & \xrightarrow{\bar{\alpha}} & G_1 \times_{c,d} G_1 & & \\ & \searrow \mathcal{F}(\alpha) & \searrow m & & \\ & & E_{J(Q)} & \xrightarrow{\bar{q}} & G_1 \\ & \searrow \sigma_P & \downarrow \sigma_Q & & \downarrow c \\ & & H_0 & \xrightarrow{Q_0} & G_0 \end{array}$$

commutes. Set theoretically, the map $\mathcal{F}(\alpha)$ sends the pair $(y, x \xrightarrow{f} p_0(y)) \in E_P$ to the pair $(y, x \xrightarrow{f} p_0(y) \xrightarrow{\alpha(y)} q_0(y)) \in E_Q$

5.7. REMARK. Equivalently, $\mathcal{F}: \text{Grpd}(\mathcal{C}) \rightarrow \mathcal{B}(\mathcal{C})$ can be obtained as the composite of $J: \text{Grpd}(\mathcal{C}) \rightarrow \text{XMod}(\mathcal{C})$ with the embedding $\mathcal{B}: \text{XMod}(\mathcal{C}) \rightarrow \mathcal{B}(\mathcal{C})$ which is the identity on objects and acts on hom-categories by the reduced composition with the identity butterfly $\cdot_{rc} I_{\mathbb{G}}: \text{XMod}(\mathcal{C})(\mathbb{H}, \mathbb{G}) \rightarrow \mathcal{B}(\mathcal{C})(\mathbb{H}, \mathbb{G})$.

5.8. THEOREM. *The homomorphism*

$$\mathcal{F}: \text{Grpd}(\mathcal{C}) \rightarrow \mathcal{B}(\mathcal{C})$$

defined in 5.6 is the bicategory of fractions of $\text{Grpd}(\mathcal{C})$ with respect to the class Σ of weak equivalences.

PROOF. Since the class Σ has a right calculus of fractions (Propositions 5.5 and 5.2 in [49]), we have to prove that \mathcal{F} satisfies conditions EF0 – EF3 of Proposition 5.2.

EF0 : Consider a weak equivalence of groupoids and the corresponding morphism $P: \mathbb{H} \rightarrow \mathbb{G}$ of crossed modules:

$$\begin{array}{ccc} H & \xrightarrow{p} & G \\ \partial \downarrow & & \downarrow \partial \\ H_0 & \xrightarrow{p_0} & G_0 \end{array}$$

As recalled in Remark 2.13 the arrows induced on kernels and cokernels of ∂ are isomorphisms. As a first step, we show that the previous diagram is a pullback. For this, consider the regular epi - mono factorizations of ∂ :

$$\begin{array}{ccc} H & \xrightarrow{p} & G \\ \partial_1 \downarrow & & \downarrow \partial_1 \\ I(H) & \xrightarrow{I(p)} & I(G) \\ \partial_2 \downarrow & & \downarrow \partial_2 \\ H_0 & \xrightarrow{p_0} & G_0 \end{array}$$

By the (normalized) Barr-Kock Theorem 9.2, the upper square is a pullback because the two regular epimorphisms ∂_1 have isomorphic kernels. As far as the lower square is concerned, observe that $\partial_2: I(H) \rightarrow H_0$ is normal (precrossed module condition in 2.4) and, therefore, it is the kernel of its cokernel. Using this fact and the fact that the arrow between cokernels is a monomorphism, it is easy to check that the lower square satisfies the universal property of the pullback.

Now, we want to show that the split butterfly

$$E_P: \mathbb{H} \rightarrow \mathbb{G}$$

associated to the above morphism of crossed modules as in 4.2 is an equivalence. Following

Proposition 3.9, it is enough to show that E_P is flippable. For this, consider the diagram

$$\begin{array}{ccc}
 H & \xrightarrow{p} & G \\
 h^\bullet \downarrow & & \downarrow g^\bullet \\
 H_1 & & G_1 \\
 \langle d, p_1 i \rangle \downarrow & \xrightarrow{\bar{p}} & \downarrow c \\
 E_P & & G_1 \\
 \sigma_P \downarrow & & \downarrow c \\
 H_0 & \xrightarrow{p_0} & G_0
 \end{array}$$

The whole square is precisely $P: \mathbb{H} \rightarrow \mathbb{G}$, so that it is a pullback. The lower square also is a pullback (4.2), so that the upper square is a pullback. From this and the fact that g^\bullet is the kernel of $d: G_1 \rightarrow G_0$, we immediately get that $h^\bullet \langle d, p_1 i \rangle$ is the kernel of $\bar{p}d: E_P \rightarrow G_0$. Finally, $\bar{p}d$ is a regular epimorphism by definition of essential surjective (2.3) and, therefore, it is the cokernel of its kernel.

EF1: Since \mathcal{F} on objects is the composite

$$Grpd(\mathcal{C}) \rightarrow XMod(\mathcal{C}) \rightarrow \mathcal{B}(\mathcal{C})$$

with the first step being an equivalence and the second one being the identity on objects, condition EF1 is clearly satisfied.

EF2: We are to prove that $\mathcal{F}: Grpd(\mathcal{C}) \rightarrow \mathcal{B}(\mathcal{C})$ is full and faithful on 2-cells. To this end, let us consider two parallel morphisms of crossed modules

$$P, Q: \mathbb{H} \rightrightarrows \mathbb{G}$$

and a morphism

$$f: E_P \rightarrow E_Q$$

between the corresponding split butterflies (4.2), i.e. the following four triangles commute:

$$\begin{array}{ccc}
 H & \xrightarrow{\langle \partial, pg^\bullet \rangle} & E_P \\
 \langle \partial, qg^\bullet \rangle \downarrow & \swarrow f & \downarrow \bar{p}d \\
 E_Q & \xrightarrow{\bar{q}d} & G_0
 \end{array}
 \quad
 \begin{array}{ccc}
 E_P & \xleftarrow{\langle 0, g \rangle} & G \\
 \sigma_P \downarrow & \swarrow f & \downarrow \langle 0, g \rangle \\
 H_0 & \xleftarrow{\sigma_Q} & E_Q
 \end{array}$$

(i) (ii) (iii) (iv)

Consider also the arrow π given by the universal property of the pullback E_P :

$$\begin{array}{ccc}
 H_0 & \xrightarrow{e} & H_1 \\
 \pi \searrow & & \downarrow p_1 \\
 & & G_1 \\
 \sigma_P \downarrow & \xrightarrow{\bar{p}} & \downarrow c \\
 H_0 & \xrightarrow{p_0} & G_0
 \end{array}$$

1

Put

$$\alpha_f: H_0 \xrightarrow{\pi} E_P \xrightarrow{f} E_Q \xrightarrow{\bar{q}} G_1$$

It is easy to check that $\alpha_f \cdot d = p_0$ and $\alpha_f \cdot c = q_0$: just use commutativity of (ii) and (iii) above. To prove that α is natural requires some computations. Following the characterization of Proposition 2.10, we are to show that

$$\begin{array}{ccccc} H & \xrightarrow{\partial} & H_0 & \xrightarrow{\pi} & E_P & \xrightarrow{f} & E_Q \\ \langle p, q \rangle \downarrow & & & & & & \downarrow \bar{q} \\ G \times G & \xrightarrow{m_0} & & & & & G_1 \end{array}$$

commutes, where $m_0 = g \sharp g^\bullet$ is the cooperator of g and g^\bullet . To this end, let us consider the following diagram, whose outer rectangle gives naturality of α_f :

$$\begin{array}{ccccc} H & \xrightarrow{\partial\pi} & E_P & \xrightarrow{f} & E_Q \\ & \searrow (1) & \uparrow \varphi_P & \nearrow \varphi_Q & \\ & & H \times G & & \\ & \swarrow \langle q, 1 \rangle & \nwarrow (3) & \searrow qg^\bullet \sharp q & \\ G \times G & \xrightarrow{\text{twist}} & G \times G & \xrightarrow{m_0} & G_1 \end{array}$$

(4)

where the maps φ_P and φ_Q are the cooperators relative to butterflies E_P and E_Q (see 5.3). The commutativity of (2), (3) and (4) is easily obtained by uniqueness of cooperators, by means of the precompositions with canonical morphisms $H \xrightarrow{\langle 1, 0 \rangle} H \times G \xleftarrow{\langle 0, 1 \rangle} G$. Observe that in proving (2) we use precisely the hypothesis (i) and (iv) above. In fact (1, f) is precisely the morphism of the spans (determined by the butterflies E_P and E_Q) corresponding to f . Finally we show that (1) commutes, by composing with pullback projections σ_P and \bar{p} :

1. $\partial \cdot \pi \cdot \sigma_P = \partial = \langle 1, p \rangle \cdot \text{pr}_1 \cdot \partial = \langle 1, p \rangle \cdot \varphi_P \cdot \sigma_P$ where the last equality is just the weak equivalence $(\text{pr}_1, \sigma_P): \varphi_P \rightarrow \partial$ in the span of crossed modules corresponding to E_P .
2. First observe that

$$\begin{array}{ccc} G & \xrightarrow{\partial e} & G_1 \\ & \searrow \langle 1, 1 \rangle & \nearrow m_0 \\ & G \times G & \end{array}$$

commutes. This can be easily deduced from the very definition of m_0 (this equation is one of the axioms defining a Peiffer graph, see [38]). Hence we start our computation:

$\partial \cdot \pi \cdot \bar{p} = \partial \cdot p_0 \cdot e = p \cdot \partial \cdot e = \langle p, p \rangle \cdot m_0 = \langle 1, p \rangle \cdot (p \times 1) \cdot m_0 = \langle 1, p \rangle \cdot \varphi_P \cdot \bar{p}$.
The last equality is obtained by observing that both $(p \times 1) \cdot m_0$ and $\varphi_P \cdot \bar{p}$ are *the* coproduct of pg and g^\bullet .

EF3: We want to prove that the diagram in $\mathcal{B}(\mathcal{C})$

$$\begin{array}{ccc} & [E] & \\ E(\pi_H, \sigma) \swarrow & & \searrow E(\pi_G, \rho) \\ \mathbb{H} & \xrightarrow{E} & \mathbb{G} \end{array}$$

commutes (up to 2-cell). For this, we use reduced composition described in 4.4, and we compute $(\pi_H, \sigma) \cdot_{rc} E =$

$$\begin{array}{ccccc} H \times G & \xrightarrow{\pi_H} & H & & G \\ \downarrow \varphi & \searrow \langle \varphi, \pi_H \kappa \rangle & \downarrow \partial & \searrow \kappa & \downarrow \partial \\ & R[\sigma] & \xrightarrow{\sigma_2} & E & \\ \swarrow \sigma_1 & & \downarrow \sigma & \swarrow \rho & \\ E & \xrightarrow{\sigma} & H_0 & & G_0 \end{array} = \begin{array}{ccccc} H \times G & \xrightarrow{\langle \varphi, \pi_H \kappa \rangle} & R[\sigma] & & G \\ \downarrow \varphi & \searrow \langle 0, \iota \rangle & \downarrow \partial & \searrow \iota & \downarrow \partial \\ & R[\sigma] & \xrightarrow{\sigma_2 \rho} & G_0 & \\ \swarrow \sigma_1 & & \downarrow \sigma & \swarrow \rho & \\ E & \xrightarrow{\sigma} & H_0 & & G_0 \end{array}$$

and $(\pi_G, \rho) \cdot_{rc} I_{\mathbb{G}} =$

$$\begin{array}{ccccc} H \times G & \xrightarrow{\pi_G} & G & & G \\ \downarrow \varphi & \searrow v & \downarrow \partial & \searrow g^\bullet & \downarrow \partial \\ & R[\sigma] & \xrightarrow{\bar{p}i} & G_1 & \\ \swarrow \sigma_1 & & \downarrow c & \swarrow g & \\ E & \xrightarrow{\rho} & G_0 & & G_0 \end{array} = \begin{array}{ccccc} H \times G & \xrightarrow{v} & R[\sigma] & & G \\ \downarrow \varphi & \searrow u & \downarrow \partial & \searrow u & \downarrow \partial \\ & R[\sigma] & \xrightarrow{\bar{p}id} & G_0 & \\ \swarrow \sigma_1 & & \downarrow d & \swarrow c & \\ E & \xrightarrow{\rho} & G_0 & & G_0 \end{array}$$

where the diagram $\sigma_1 \cdot \rho = \bar{p}i \cdot c$ is a pullback as far as the diagram $\sigma_1 \cdot \rho = \bar{p} \cdot d$ is (compare with the diagram in 3.5). We are going to prove that the two butterflies obtained this way are isomorphic. In fact they coincide.

We see that $\bar{p}id = \bar{p}c = \sigma_2 \rho$ commutes as it is one of the pullbacks in the diagram of Remark 3.5. To show that $\langle 0, \iota \rangle = u$ it suffices to compose the last with the pullback projections σ_1 and σ_2 that define $\langle 0, \iota \rangle$: $u\sigma_1 = 0$ by definition, $u\sigma_2$ is the normalization of the groupoid $(R[\sigma], \sigma_1, \sigma_2)$, hence the morphism ι .

In conclusion we prove that $\langle \varphi, \pi_H \kappa \rangle = v$. For this it suffices to compose with the second projection $\bar{p}i$, since the first is identical. Hence we consider the following commutative

diagrams:

$$\begin{array}{ccc}
 H \times G & \xrightarrow{\pi_H} & H \\
 \langle \varphi, \pi_H \kappa \rangle \downarrow & (4) & \downarrow \kappa \\
 R[\sigma] & \xrightarrow{\sigma_2} & E \\
 \sigma_1 \downarrow & (5) & \downarrow \sigma \\
 E & \xrightarrow{\sigma} & H_0
 \end{array}
 \qquad
 \begin{array}{ccc}
 H \times G & \xrightarrow{\pi_H} & H \\
 \langle \varphi, \pi_H \kappa \rangle \downarrow & (4) & \downarrow \kappa \\
 R[\sigma] & \xrightarrow{\sigma_2} & E \\
 \bar{\rho} \cdot i \downarrow & (6) & \downarrow \rho \\
 G_1 & \xrightarrow{d} & G_0
 \end{array}$$

Since $\langle \varphi, \pi_H \kappa \rangle \sigma_1 = \varphi$ and $\kappa \sigma = \partial$, then (4) + (5) is a pullback. Since (5) is a pullback and σ_2 is a split epimorphism, (4) is a pullback. But (6) also is a pullback, so that (4) + (6) is a pullback. Since $\kappa \rho = 0$ and $gi = g^\bullet$ is the kernel of d , (4) + (6) can be written as

$$\begin{array}{ccc}
 H \times G & \xrightarrow{\pi_H} & H \\
 \pi_G \downarrow & & \downarrow \\
 G & \longrightarrow & 0 \\
 g^\bullet \downarrow & & \downarrow \\
 G_1 & \xrightarrow{d} & G_0
 \end{array}$$

In particular, this gives $\langle \varphi, \pi_H \kappa \rangle \bar{\rho} i = \pi_G g^\bullet$, as requested. \blacksquare

5.9. REMARK. Condition EF3 above gives an alternative proof of the construction of section 4.3. Moreover it yields a recipe to obtain the crossed modules morphism corresponding to a split butterfly.

Let us consider the span associated to the split butterfly $E = (\kappa, \iota, \sigma, \rho)$, and suppose we have chosen a section s of the split epimorphism σ . Then, since the left leg of the span is a pullback diagram, we can pull the section s back along φ , and get the morphism of crossed modules (\bar{s}, s) , moreover the last is a section of (π_H, σ) . The situation is summarized in the diagram below.

$$\begin{array}{ccccc}
 H & \xrightleftharpoons[\pi_H]{\bar{s}} & H \times G & \xrightarrow{\pi_G} & G \\
 \partial \downarrow & & \varphi \downarrow & & \downarrow \partial \\
 H_0 & \xrightleftharpoons[\sigma]{s} & E & \xrightarrow{\rho} & G_0
 \end{array}$$

Then we can compose on the left the equation $E_{(\pi_H, \sigma)} E = E_{(\pi_G, \rho)}$ with the morphism (\bar{s}, s) and get:

$$\begin{aligned}
 E &= (\bar{s}, s)(\pi_H, \sigma) \cdot_{rc} E = (\bar{s}, s) \cdot_{rc} E_{(\pi_H, \sigma)} E = (\bar{s}, s) \cdot_{rc} E_{(\pi_G, \rho)} = \\
 &= (\bar{s}, s)(\pi_G, \rho) \cdot_{rc} I_G = (\bar{s} \pi_G, s \rho) \cdot_{rc} I_G = E_{(\bar{s} \pi_G, s \rho)}
 \end{aligned}$$

i.e. the morphism $(\bar{s} \pi_G, s \rho)$ is associated with the (split) butterfly E . Notice the arbitrary choice of the section s : if another section is chosen, the construction yields another representant of E in the same 2-isomorphism class.

6. From butterflies to weak morphisms: three concrete examples

In the following we show how to construct the weak morphism associated to a butterfly in the cases of groups, Lie algebras and Rings (but the technique can be adapted to other semi-abelian algebraic varieties), and we give an idea of how to recover a butterfly from a weak morphism. The first two instances are well known in the literature: a butterfly between two crossed modules of groups corresponds to a weak morphism of strict 2-groups (from Theorem 5.8 and [49]), similarly a butterfly in Lie algebras gives a homomorphism (i.e. semi-strict morphism) of strict Lie 2-algebras (see [40] and [42]). In a similar way, a butterfly in rings provides the data for a weak morphism of strict 2-rings. This notion seems not to be present in the literature, although there are two notions of weak 2-rings with units (the categorical rings of [34] and the Ann-categories of [45]). These two notions coincide in the strict case, and it is possible to show that our weak morphism of strict 2-rings specializes to those.

6.1. THE TECHNIQUE. Let us consider the butterfly $E = (E, \kappa, \rho, \iota, \sigma)$ in a semi-abelian algebraic variety \mathcal{C} , and let $U : \mathcal{C} \rightarrow \mathcal{S}$ (the axiom of choice holding in \mathcal{S}) a suitable functor that forgets part of the structure. Let s be a section of $U(\sigma)$.

$$\begin{array}{ccccc}
 H & & & & G \\
 \downarrow \partial & \searrow \kappa & & \swarrow \iota & \downarrow \partial \\
 & & E & & \\
 & \nearrow s & & \searrow \rho & \\
 H_0 & & & & G_0
 \end{array}$$

We want to show how E yields a weak morphism of groupoids $F_E : \mathbb{H} \rightarrow \mathbb{G}$. From now on we will write just σ for $U(\sigma)$, etc.

The functor U preserves finite limits, so that it extends to a 2-functor between the 2-categories of internal groupoids. Now, to the butterfly E is associated a span (see section 5.5) in $Grpd(\mathcal{C})$ with the left leg being a weak equivalence:

$$\mathbb{H} \xleftarrow{S} [E] \xrightarrow{R} \mathbb{G},$$

more explicitly:

$$\begin{array}{ccccc}
 H_1 & \xleftarrow{\pi_H \rtimes \sigma} & (H \times G) \rtimes_{\bar{\xi}} E & \xrightarrow{\pi_G \rtimes \rho} & G_1 \\
 d \downarrow c & & d \downarrow c & & d \downarrow c \\
 H_0 & \xleftarrow{\sigma} & E & \xrightarrow{\rho} & G_0
 \end{array}$$

By applying U to this construction, S turns in an equivalence in $Grpd(\mathcal{S})$, so that it has a weak inverse S^* .

The composition S^*R (which is an internal functor in $Grpd(\mathcal{S})$) is a good candidate for a weak morphism in $Grpd(\mathcal{C})$, with the coherence conditions encoded in the short exact sequence of the butterfly.

6.2. CASE STUDY: GROUPS. Let $\mathcal{C} = \text{Grp}$, and $U : \text{Grp} \rightarrow \text{Set}_*$ the underlying pointed-set functor.

Under the equivalence between crossed modules and groupoids, the crossed module $\partial : H \rightarrow H_0$ gives rise to the groupoid in groups

$$G_1 \begin{array}{c} \xrightarrow{d} \\ \xleftarrow{c} \\ \xrightarrow{c} \end{array} G_0$$

where G_1 is the semidirect product $G \rtimes G_0$, and structure maps result (additive notation)

$$c : (g, x) \mapsto x, \quad d : (g, x) \mapsto \partial g + x, \quad e : x \mapsto (0, x).$$

Define the monoidal functor $F_E = (F_0, F_1, F_2)$:

$$\begin{aligned} F_0 &= s\rho : H_0 \rightarrow G_0; & x &\mapsto \rho(sx) \\ F_1 &= F \rtimes F_0 & \text{where} \\ & F : H \rightarrow G; & h &\mapsto -\kappa(h) + s(\partial(h)) \\ F_2 &: H_0 \times H_0 \rightarrow G_1; & (x, y) &\mapsto (sx + sy - s(x + y), \rho(s(x + y))) \end{aligned}$$

Notice that, since $\partial(sx + sy - s(x + y)) + \rho(s(x + y)) = \rho((sx + sy - s(x + y))) + \rho(s(x + y)) = \rho(sx) + \rho(sy)$, $F_2(x, y)$ is to be interpreted as an arrow $F_0(x + y) \rightarrow F_0(x) + F_0(y)$.

From the classification of group extensions (see [ML Homology], for instance) we know that with the short exact sequence

$$G \xrightarrow{\kappa} E \xrightarrow{\sigma} H_0$$

with a chosen set-theoretical section s of σ we can associate two of functions $\alpha : H_0 \rightarrow \text{Aut} G$ and $f : H_0 \times H_0 \rightarrow G$: with $\alpha(x)(g) = x \cdot g = sx + g - sx$ and $f(x, y) = sx + sy - s(x + y)$. Such functions satisfy the following well known relation: for any x, y, z in H_0

$$x \cdot f(y, z) + f(x, yz) = f(x, y) + f(xy, z).$$

It is now easy to show that this relation corresponds precisely to what is necessary in order to prove (associative) coherence for the monoidal functor F_E .

A further remark. In giving the example above we stressed the role of group extensions classification. This is surely of interest, but it is not conceptually necessary. Actually, the reason why we can extend the method used in Remark 5.9, is that the monoidal functor S above, being a monoidal equivalence, has a weak inverse.

Finally we give a glance to the construction of a butterfly from a (normalized) monoidal functor of 2-groups. Consider, $F = (F_0, F_1) : \mathbb{H} \rightarrow \mathbb{G}$ a functor with monoidal structure isomorphisms $F_2^{x_1, x_2} : F_0x_1 + F_0x_2 \rightarrow F_0(x_1 + x_2)$. Define P_0 as the following limit (in Set)

$$\begin{array}{ccccc} & & P_0 & & \\ & \swarrow \sigma & \downarrow & \searrow \rho & \\ H_0 & & G_1 & & G_0 \\ & \searrow F_0 & \swarrow c & \searrow d & \swarrow 1 \\ & G_0 & & G_0 & \end{array}$$

i.e.

$$P_0 = \{(y, g, x) \in H_0 \times G_1 \times G_0 \text{ s.t. } F_0 y = cg \text{ and } dg = x\}$$

and

$$\sigma: (y, g, x) \mapsto y, \quad \rho: (y, g, x) \mapsto x$$

Despite the fact that F_0 is not a group homomorphism, it can be proved that P_0 is a group and σ and ρ are group homomorphisms (see [49], Proposition 6.3). Moreover, $\ker \sigma = \ker c$ (just because c is surjective). Finally, the factorization $\kappa: H \rightarrow P_0$ of the commutative diagram

$$\begin{array}{ccccc}
 & & H & & \\
 & \swarrow^{hd} & \downarrow^{hiF_1} & \searrow^0 & \\
 H_0 & & G_1 & & G_0 \\
 & \searrow^{F_0} & \swarrow^c & \searrow^d & \swarrow^1 \\
 & & G_0 & & G_0
 \end{array}$$

through the limit P_0 is also a group homomorphism (this follows immediately from the naturality of the monoidal structure of F). We get in this way the required butterfly $(P_0, \kappa, \rho, \iota, \sigma)$.

6.3. CASE STUDY: LIE ALGEBRAS. A groupoid in *Lie* is called a strict Lie 2-Algebra in [3]. Now we consider the forgetful functor $U: Lie \rightarrow Vect$. As for the case of groups, it extends to internal groupoids. With notation as above we can define F_E with the same technique. Indeed F_0 and F_1 are defined in the same way (provided the semidirect product is performed in *Lie!*), while

$$F_2: (x, y) \mapsto ([sx, sy] - s[x, y], \rho(s[x, y])).$$

From the theory of Lie algebras extensions, we know that with the extension (ι, σ) (and a chosen linear section s of σ) is associated a linear map $\alpha: H_0 \rightarrow DerG$, $\alpha(x)(g) = x \cdot g = [sx, g]$, and a bilinear skew-symmetric map $f: H_0 \times H_0 \rightarrow G$, $f(x, y) = [sx, sy] - s[x, y]$. These maps satisfy the relations

$$(i) \text{ for any } x, y \text{ in } H_0, [\alpha(x), \alpha(y)] - \alpha([x, y]) = \text{ad}_{f(x, y)}$$

$$(ii) \text{ for any } x, y, z \text{ in } H_0$$

$$\sum_{\text{cyclic}} (x \cdot f(y, z) - f([x, y], z)) = 0$$

where ad_g is the (adjoint) action defined by $\text{ad}_g(g') = [g, g']$. Computations show that the first relation helps in proving the naturality of F_2 , while the second relation yields the coherence of the bracket operation with respect to the jacobian identity.

6.4. **CASE STUDY: RINGS.** We call (strict) 2-*ring* a groupoid in the category of rings. There are two obvious forgetful 2-functors from the 2-categories of (strict) categorical rings [34] and of (strict) Ann-categories [45] respectively, both with strict homomorphisms and 2-homomorphisms as 1-cells and 2-cells.

We consider the forgetful functor $U: Rng \rightarrow Set_*$, that, as for the previous cases, extends to groupoids. The definition of F_E goes verbatim as in the case of groups, thanks to the additive notation used there, that now expresses the underlying abelian group structure of a ring.

Again, the exact sequence (ι, σ) provides the data for proving that F_E is a 2-ring homomorphism. In fact we use s , the set-theoretical section of σ , to define $f, \epsilon: H_0 \times H_0 \rightarrow G$: $f(x, y) = sx + sy - s(x + y)$, $\epsilon(x, y) = sx \cdot sy - s(x \cdot y)$, and a map $\alpha: H_0 \rightarrow \text{Bim}G$ with $\alpha(x)(g) = (sx \cdot g, g \cdot sx)$. Then the following relations hold for any x, y, z and t in H_0 (see [36]):

- (i) $\alpha(x) + \alpha(y) - \alpha(x + y) = \mu_{f(x, y)}$
- (ii) $\alpha(x) \circ \alpha(y) - \alpha(xy) = -\mu_{\epsilon(x, y)}$
- (iii) $f(0, y) = 0 = f(x, 0)$ and $\epsilon(0, y) = 0 = \epsilon(x, 0)$
- (iv) $f(x, y) + f(z, t) - f(x + z, y + t) - f(x, z) - f(y, t) + f(x + y, z + t) = 0$
- (v) $-\epsilon(x, t) - \epsilon(y, t) + \epsilon(x + y, t) + f(xt, yt) - f(x, y) \cdot t = 0$
- (vi) $\epsilon(t, x) + \epsilon(t, y) - \epsilon(t, x + y) - f(tx, ty) + f \cdot h(x, y) = 0$
- (vii) $x \cdot \epsilon(y, z) - \epsilon(xy, z) + \epsilon(x, yz) - \epsilon(x, y) \cdot z = 0$

where μ_g is the inner bimultiplication induced by the multiplication with g .

Now, (i) and (ii) give the naturality of F_2 . Moreover, since the normalization conditions (iii) hold, the relation (iv) gives at once associative and symmetric coherence: actually for $y = 0$ we obtain the cocycle condition for the underlying (abelian) group extension, while letting $x = t = 0$ we get the symmetric coherence. Finally (vii) yields the associative coherence for the multiplication, and (v) and (vi) give the distributive coherence.

6.5. **REMARK.** If \mathcal{C} is a category with finite limits (not necessarily semi-abelian), Theorem 5.8 may fail. In this more general case, internal categories may differ from internal groupoids and the bicategory of fractions

$$Cat(\mathcal{C})[\Sigma^{-1}]$$

may still admit a (more involved) explicit description: it is the bicategory of internal anafunctors. This has been proved independently by M. Dupont in [23], where the base category \mathcal{C} is assumed to be regular, and by D. Roberts in [46], where essential surjectivity is intended relatively to a Grothendieck topology on \mathcal{C} and internal categories (not only internal groupoids) are considered.

7. Classification of extensions

In this section we assume that \mathcal{C} has split extensions classifiers (see [14], and section 9), as it happens, for instance, in the category of groups or of Lie-algebras. Consider two objects H and G in \mathcal{C} . Let $D(H) = (0 \rightarrow H)$ be the discrete crossed module on H and

$$\mathcal{A}(G) = (\mathcal{I}_G: G \rightarrow \text{Aut}G, \text{ev}: \text{Aut}G \wr G \rightarrow G)$$

the crossed module associated with the split extensions classifier $\text{Aut}G$ (that is, the crossed module corresponding to the action groupoid).

The following lemma generalizes Example 13.4 of [40].

7.1. LEMMA. *The groupoid*

$$\text{Ext}(H, G)$$

of extensions of the form $H \leftarrow E \leftarrow G$ is isomorphic to the groupoid

$$\mathcal{B}(\mathcal{C})(D(H), \mathcal{A}(G))$$

Such an isomorphism restricts to split extensions and split butterflies.

PROOF. Let us start with a butterfly

$$\begin{array}{ccccc} 0 & & & & G \\ & \searrow ! & & \swarrow \iota & \downarrow \mathcal{I}_G \\ & & E & & \\ & \swarrow \sigma & & \searrow \rho & \\ H & & & & \text{Aut}G \end{array}$$

We are going to prove that ρ is uniquely determined. Following Remark 3.2, the right wing determines a discrete fibration of groupoids. Hence the diagram

$$\begin{array}{ccc} G \rtimes_{(\rho \flat 1) \text{ev}} E & \xrightarrow{p_E} & E \\ 1 \rtimes \rho \downarrow & (*) & \downarrow \rho \\ G \rtimes_{\text{ev}} \text{Aut}G & \xrightarrow{p_{\text{Aut}G}} & \text{Aut}G \end{array}$$

is a pullback. Therefore, ρ is the unique arrow making the diagram above a pullback (universal property of $\text{Aut}G$).

Conversely, consider an extension

$$H \xleftarrow{\sigma} E \xleftarrow{\iota} G$$

Since ι is normal in E , there exists a unique χ_ι such that

$$\begin{array}{ccc} E \wr G & \xrightarrow{\chi_\iota} & G \\ 1 \wr \iota \downarrow & & \downarrow \iota \\ E \wr E & \xrightarrow{\chi_E} & E \end{array}$$

commutes. By the universal property of $\text{Aut}G$, we get a unique ρ such that diagram (*) above is a pullback and

$$\begin{array}{ccc} E \wr G & \xrightarrow{\chi|} & \text{Aut}G \wr G \\ \rho \wr 1 \searrow & & \nearrow \text{ev} \\ & G & \end{array}$$

commutes. It remains to show that the extension (ι, σ) , equipped with ρ , is a butterfly from $D(H)$ to $\mathcal{A}(G)$. Condition 3.1.iv follows by pasting together the diagram (*) with the following one:

$$\begin{array}{ccccc} G \rtimes_{\chi_G} G & \xrightarrow{P_G} & G & & \\ 1 \rtimes \iota \downarrow & & (** & & \downarrow \iota \\ G \rtimes_{(\rho \wr 1) \text{ev}} E & \xrightarrow{p_E} & E & & \end{array}$$

As far as the commutativity of the right wing is concerned, observe that diagram $(**)+(*)$ is a pullback (because both $(*)$ and $(**)$ are pullbacks) and use once again the universal property of $\text{Aut}G$ in order to conclude that $\iota\rho = \mathcal{I}_G$. ■

7.2. CLASSIFICATION. Combining the previous isomorphism of groupoids with Theorem 5.8, we get a very general classification of extensions:

$$\text{Ext}(H, G) \simeq \mathcal{B}(\mathcal{C})(D(H), \mathcal{A}(G)) \simeq \text{Grpd}(\mathcal{C})[\Sigma^{-1}](D(H), \mathcal{A}(G))$$

To recover the classical classification of group extensions in terms of factor sets due to Schreier it suffices to use another result from [49] already quoted in the Introduction: when \mathcal{C} is the category of groups, the bicategory of fractions $\text{Grpd}(\mathcal{C})[\Sigma^{-1}]$ can be described as the bicategory of groupoids, monoidal functors and monoidal natural transformations, and a monoidal functor from $D(H)$ to $\mathcal{A}(G)$ is nothing but a factor set.

8. The free exact case

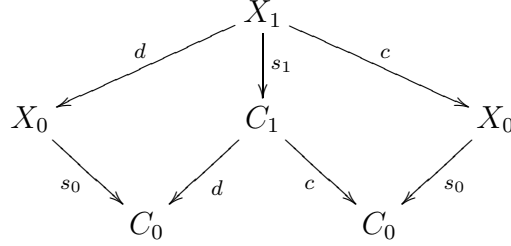
When \mathcal{C} is the category of groups, the main result of [40] is not stated in terms of bicategory of fractions, but it is stated as an equivalence of groupoids

$$\mathcal{B}(\mathcal{C})(\mathbb{H}, \mathbb{G}) \simeq X\text{Mod}(\mathcal{C})(\mathbb{K}, \mathbb{G})$$

where \mathbb{K} is the crossed module of groups obtained from \mathbb{H} by pulling back $\partial: H \rightarrow H_0$ along a surjective homomorphism $K_0 \rightarrow H_0$ with K_0 being a free group. The same is done for Lie algebras in [1]. The aim of this section is to generalize the previous equivalence to the case when the base category \mathcal{C} is also free exact.

8.1. FREE EXACT CATEGORIES. We assume that the semi-abelian category \mathcal{C} is free exact in the sense of [21], that is, it has enough regular projective objects. This means that for every object X in \mathcal{C} there exists a regular epimorphism $x: X' \rightarrow X$ with X' regular projective. All semi-abelian varieties are of this kind. In particular, groups and Lie algebras are free exact semi-abelian categories.

8.2. **REPLACEMENT.** Let \mathbb{C} be a groupoid and $s_0: X_0 \rightarrow C_0$ a regular epimorphism with X_0 regular projective. Consider the limit



The internal graph $d, c: X_1 \rightrightarrows X_0$ inherits a structure of groupoid from that of \mathbb{C} . Moreover, the internal functor $s = (s_1, s_0): \mathbb{X} \rightarrow \mathbb{C}$ is a weak equivalence (it is full and faithful by construction of s_1 , and it is essentially surjective because s_0 is a regular epimorphism). Finally, observe that, since X_0 is regular projective, the groupoid \mathbb{X} is Σ -projective: every weak equivalence with codomain \mathbb{X} is in fact an equivalence.

8.3. **PROPOSITION.** *Let \mathbb{C} and \mathbb{D} be groupoids and fix a replacement $s: \mathbb{X} \rightarrow \mathbb{C}$ as in 8.2. There is an equivalence of groupoids*

$$\mathcal{B}(\mathcal{C})(J(\mathbb{C}), J(\mathbb{D})) \simeq \text{Grpd}(\mathcal{C})(\mathbb{X}, \mathbb{D})$$

PROOF. Since $s: \mathbb{X} \rightarrow \mathbb{C}$ is a weak equivalence, $\mathcal{F}(s): J(\mathbb{X}) \rightarrow J(\mathbb{C})$ is an equivalence (see condition EF0 in the proof of Theorem 5.8). Therefore, $\mathcal{F}(s)$ induces an equivalence

$$\mathcal{B}(\mathcal{C})(J(\mathbb{C}), J(\mathbb{D})) \simeq \mathcal{B}(\mathcal{C})(J(\mathbb{X}), J(\mathbb{D}))$$

Moreover, since X_0 is regular projective, all extensions of the form $X_0 \leftarrow E \leftarrow D$ split and then all butterflies from $J(\mathbb{X})$ to $J(\mathbb{D})$ split:

$$\mathcal{B}(\mathcal{C})(J(\mathbb{X}), J(\mathbb{D})) = \mathcal{B}(\mathcal{C})(J(\mathbb{X}), J(\mathbb{D}))_{\text{split}}$$

Finally, following 4.3, we have

$$\mathcal{B}(\mathcal{C})(J(\mathbb{X}), J(\mathbb{D}))_{\text{split}} \simeq \text{Grpd}(\mathcal{C})(\mathbb{X}, \mathbb{D})$$

■

8.4. **REMARK.** To end this section, we sketch a general argument on bicategories of fractions which subsumes Proposition 8.3.

Let Σ be a class of 1-cells in a bicategory \mathcal{B} with a right calculus of fractions. Assume that:

1. Σ satisfies the 2 \Rightarrow 3 property: let $F: \mathbb{C} \rightarrow \mathbb{D}$ and $G: \mathbb{D} \rightarrow \mathbb{E}$ be 1-cells in \mathcal{B} ; if two of F , G and $F \cdot G$ are in Σ , then the third one is in Σ ;
2. Every 1-cell $W: \mathbb{C} \rightarrow \mathbb{D}$ of Σ is full and faithful, that is, for all objects \mathbb{A} the functor $\mathcal{B}(\mathbb{A}, W): \mathcal{B}(\mathbb{A}, \mathbb{C}) \rightarrow \mathcal{B}(\mathbb{A}, \mathbb{D})$ is full and faithful;

3. For every object \mathbb{C} in \mathcal{B} there exists $S: \mathbb{X} \rightarrow \mathbb{C}$ in Σ with \mathbb{X} a Σ -projective object.

Then, if we fix objects \mathbb{C} and \mathbb{D} and a 1-cell $S: \mathbb{X} \rightarrow \mathbb{C}$ as in 3, the functor assigning to a 1-cell $F: \mathbb{X} \rightarrow \mathbb{D}$ the span

$$\begin{array}{ccc} & \mathbb{X} & \\ S \swarrow & & \searrow F \\ \mathbb{C} & & \mathbb{D} \end{array}$$

yields an equivalence of groupoids

$$\mathcal{B}(\mathbb{X}, \mathbb{D}) \simeq \mathcal{B}[\Sigma^{-1}](\mathbb{C}, \mathbb{D})$$

9. A reminder on semi-abelian categories

The general context where the theory of crossed modules and weak maps can be developed along the lines described in this paper is that of semi-abelian categories where an additional assumption is made: the commutativity (in the sense of Smith) of internal equivalence relations is determined by the commutativity (in the sense of Huq) of their *zero*-classes. In the following, the basic notions are recalled and the notations are fixed, for the reader's convenience.

9.1. PROTOMODULAR AND SEMI-ABELIAN CATEGORIES. Semi-abelian categories were introduced in 2002 [32], and they represent the *state-of-the-art* in the long-lasting investigations whose aim is to provide an abstract categorical setting for non (necessarily) commutative pointed algebraic structures, such as groups, rings or Lie algebras.

A category is semi-abelian when it is pointed (i.e. $0 = 1$) with finite coproducts, protomodular [12] and exact (in the sense of Barr).

Pointed protomodular categories can be characterized as pointed, finitely complete categories where the *split short five lemma* holds: given the diagram

$$\begin{array}{ccccc} K & \xrightarrow{k} & A & \begin{array}{c} \xrightarrow{p} \\ \xleftarrow{s} \end{array} & B \\ h \downarrow & & \downarrow f & & \downarrow g \\ K' & \xrightarrow{k'} & A' & \begin{array}{c} \xrightarrow{p'} \\ \xleftarrow{s'} \end{array} & B' \end{array}$$

with $kf = hk'$, $pg = fp'$ and $sf = gs'$, the morphism f is an isomorphism if k and g are.

Recall that a category is exact (in the sense of Barr) when it is *regular* and internal equivalence relations are *effective*, i.e. kernel pairs.

Finally a *regular category* is a finitely complete category where effective equivalence relations have coequalizers that are stable under pullbacks.

The protomodularity condition can be reformulated when the category \mathcal{C} is pointed regular. This is stated by the following useful characterization, that we quote from [18], IV.4.A:

9.2. (NORMALIZED) BARR-KOCK THEOREM. *Let \mathcal{C} be a regular pointed category. Then \mathcal{C} is protomodular iff the “normalized Barr-Kock” property holds: in any commutative diagram*

$$\begin{array}{ccccc} K & \xrightarrow{k} & A & \xrightarrow{p} & B \\ h \downarrow & & \downarrow f & (1) & \downarrow g \\ K' & \xrightarrow{k'} & A' & \xrightarrow{p'} & B' \end{array}$$

where $K = \ker p$, $K' = \ker p'$ and with p being regular epi,

h is an iso iff the square (1) is a pullback.

9.3. THE “HUQ = SMITH” CONDITION. In order to introduce the so-called “Huq = Smith” condition, we first recall the notions of commuting subobjects and commuting equivalence relations.

Two subobjects $G \xrightarrow{g} E \xleftarrow{h} H$ commute in the sense of Huq (see [29, 17]) if they cooperate as morphisms, i.e. if there exists a (unique) morphism φ such that the diagram

$$\begin{array}{ccccc} G & \xrightarrow{\langle 1, 0 \rangle} & G \times H & \xleftarrow{\langle 0, 1 \rangle} & H \\ & \searrow g & \downarrow \varphi & \swarrow h & \\ & & E & & \end{array}$$

commutes.

Suppose that the maps g, h are normal monomorphisms, i.e. kernels. Then the denormalized version of the above notion is that of commuting equivalence relations. A pair of equivalence relations on a common object E

$$\begin{array}{ccccc} R & \xrightleftharpoons[r_2]{r_1} & E & \xrightleftharpoons[s_2]{s_1} & S \\ & \xleftarrow{e_R} & & \xrightarrow{e_S} & \end{array}$$

commutes (in the sense of Smith, see [47, 43]) when there exists a (unique) morphism Φ such that the diagram

$$\begin{array}{ccccc} R & \xrightarrow{\langle 1, r_1 e_S \rangle} & R \times_{r_1, s_0} S & \xleftarrow{\langle s_0 e_R, 1 \rangle} & S \\ & \searrow r_0 & \downarrow \Phi & \swarrow s_1 & \\ & & E & & \end{array}$$

commutes. It is a well known fact that when two equivalence relations commute, so do their normalizations (see [17]).

The converse does not hold in general, not even in semi-abelian categories (see [13] for a counterexample, due to G. Janelidze, in the semi-abelian category of digroups). Nevertheless it does hold in several important algebraic contexts, as pointed *strongly*

protomodular categories (see [17], section 6) and pointed *action accessible* categories (see [38]). As a matter of fact, for internal structures in (many) pointed algebraic varieties, this is quite a crucial notion and it recaptures the feeling that a local behavior near the identity element determines a global behavior. Furthermore, it has been acknowledged in [31] that this property is a candidate to become an axiom for “good” semi-abelian categories. In present work we will refer to it as to the “Huq = Smith” property.

9.4. **REMARK.** Butterflies were originally defined by B. Noohi for crossed modules of groups [40], but the author himself, in [2] with E. Aldrovandi, extends the construction to crossed modules of internal groups in the Grothendieck topos $\widehat{\mathbf{S}}$, i.e. the topos of the sheaves over a site (\mathbf{S}, J) with subcanonical topology J .

As a matter of fact, the present setting generalizes the one of [2]. In fact more is true: our results apply to any pointed strongly protomodular algebraic theory in a Grothendieck topos. To see this it is necessary to recollect some results from the literature. First, in [11], Example 4.6.3 shows that if \mathbb{T} is a pointed protomodular algebraic theory, and \mathcal{C} is a regular (exact) category, then the category $\text{Alg}_{\mathbb{T}}(\mathcal{C})$ of the models of \mathbb{T} in \mathcal{C} is homological (exact homological). Hence, if \mathcal{C} is exact, the missing condition for $\text{Alg}_{\mathbb{T}}(\mathcal{C})$ to be semi-abelian is its finite cocompleteness.

Indeed the category of models of an algebraic theory in an elementary topos \mathcal{E} is finitely cocomplete if (i) the topos has a Natural Number Object, and (ii) the theory is finitely presented. Back to the situation considered here, for a Grothendieck topos \mathcal{E} , condition (i) is free, and condition (ii) can be dropped (see [11] again, the discussion after the cited example), so that $\text{Alg}_{\mathbb{T}}(\mathcal{E})$ is semi-abelian.

Concerning the condition “Huq = Smith”, strongly protomodular semi-abelian (i.e. strongly semi-abelian) categories have this nice property, and we know from [9] that for a strongly protomodular (not necessarily pointed) theory \mathbb{T} , and a finitely complete category \mathcal{E} , the category of models $\text{Alg}_{\mathbb{T}}(\mathcal{E})$ is still strongly protomodular. This is clearly the case for a Grothendieck topos \mathcal{E} .

In conclusion we can state that not only our constructions and results apply to the situation described in [2], but also in the context of internal Lie algebras, internal rings and other strongly semi-abelian theories defined internally in a Grothendieck topos \mathcal{E} .

9.5. **REMARK.** Two morphisms cooperate if their images do, and this happens precisely when their commutator is trivial, for a suitable notion of commutator. Unfortunately, describing the many aspects of the commutator theory involved would take us far beyond our purposes. The interested reader may refer to [38], and the bibliography therein.

9.6. **INTERNAL OBJECT ACTIONS.** Diverse notions of actions exist in many algebraic contexts. Most of them share the disadvantage of not being defined intrinsically, but as set-theoretical maps satisfying certain properties. From an algebraic-categorical point of view this is not convenient, since those maps are difficult to deal with. This issue has been fixed by the notion of *internal action* [16, 10], that expresses its full classifying power in the broad context of semi-abelian categories.

Let \mathcal{C} be a finitely complete pointed category with coproducts. Then for any object B in \mathcal{C} one can define a functor “ker” from the category of split epimorphisms (points) over B into \mathcal{C}

$$\ker : Pt_B(\mathcal{C}) \rightarrow \mathcal{C}, \quad \begin{array}{c} A \\ s \uparrow \downarrow p \\ B \end{array} \mapsto \ker(p).$$

This has a left adjoint:

$$B + (-) : \mathcal{C} \rightarrow Pt_B(\mathcal{C}), \quad X \mapsto \begin{array}{c} B + X \\ i_B \uparrow \downarrow [1,0] \\ B \end{array},$$

The monad corresponding to this adjunction is denoted by $Bb(-) : \mathcal{C} \rightarrow \mathcal{C}$, and, for any object A of \mathcal{C} , we obtain a kernel diagram:

$$BbA \xrightarrow{j_{B,A}} B + A \xrightarrow{[1,0]} B.$$

The $Bb(-)$ -algebras are called internal B -actions in \mathcal{C} .

Let us observe that in the case of groups, the object BbA is the group generated by the *formal conjugates* of elements of A by elements of B , i.e. by the triples of the kind (b, a, b^{-1}) with $b \in B$ and $a \in A$.

For any object A of \mathcal{C} , one can define a canonical conjugation action of A on A itself given by the composition:

$$\chi_A : AbA \xrightarrow{j_{A,A}} A + A \xrightarrow{[1,1]} A.$$

In the category of groups, the morphism χ_A is the internal action associated to the usual conjugation in A : the realization morphism $[1, 1]$ of above makes the formal conjugates of AbA computed effectively in A . Finally observe that conjugation actions are components of a natural transformation $\chi : (-)b(-) \Rightarrow \text{Id}_{\mathcal{C}}$.

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